

# SPECTRAL PROPERTIES OF THE MASSLESS RELATIVISTIC HARMONIC OSCILLATOR

József Lörinczi

*School of Mathematics, Loughborough University*  
*Loughborough LE11 3TU, United Kingdom*  
J.Lorinczi@lboro.ac.uk

Jacek Małecki

*Institute of Mathematics and Computer Science*  
*Wrocław University of Technology*  
*Wyb. Wyspiańskiego 27, 50-370 Wrocław, Poland*  
jacek.malecki@pwr.wroc.pl

## Abstract

The spectral properties of the pseudo-differential operator  $(-d^2/dx^2)^{1/2} + x^2$  are analyzed by a combination of functional integration methods and direct analysis. We obtain a representation of its eigenvalues and eigenfunctions, prove precise asymptotic formulae, and establish various analytic properties. We also derive trace asymptotics and heat kernel estimates.

# 1 Introduction

Stochastic methods based on functional integration applied to the study of properties of pseudo-differential operators and related semigroups offer a powerful alternative to the techniques of analysis [3, 8]. Typical problems addressed include spectral properties of the operator, heat kernel estimates,  $L^p$ -boundedness, ultracontractivity properties, and the decay of the eigenfunctions.

In the paper [10] we study analytic properties of evolution semigroups generated by fractional Schrödinger operators

$$H_\alpha = (-\Delta)^{\alpha/2} + V, \quad 0 < \alpha < 2$$

with (fractional) Kato-class potentials  $V$ . As  $\alpha \neq 2$  these operators generate non-Gaussian  $\alpha$ -stable processes running under the potential  $V$ . These are Lévy processes with paths having jump discontinuities. The well-known case  $\alpha = 2$  corresponds to standard Schrödinger operators generating Brownian motion in the presence of  $V$  (which is an Itô diffusion under extra conditions on the potential). The properties of Schrödinger operators and fractional Schrödinger operators in many aspects markedly differ. One sharp contrast appears in the decay properties of their ground state (first eigenfunction). Specifically, the ground state of a Schrödinger operator with pinning potential  $V(x) \rightarrow \infty$  as  $|x| \rightarrow \infty$  decays (super)exponentially while it decays only polynomially in the case of fractional Schrödinger operators. This difference is due to the heavy tails of stable processes, as opposed to the Gaussian tail of Brownian motion. Another remarkable difference is that, roughly, the Schrödinger semigroups  $e^{-t(-\Delta+V)}$  are intrinsically ultracontractive for super-quadratically increasing potentials, while this property holds for fractional Schrödinger semigroups  $e^{-t((-\Delta)^{\alpha/2}+V)}$  already for potentials increasing faster than logarithmically.

Our aim in the present paper is to focus on one single pseudo-differential operator and derive fine details on its spectrum and eigenfunctions by using a combination of functional integration and hands-on analytic methods. We will consider the operator

$$H = \sqrt{-\frac{d^2}{dx^2} + x^2}$$

and obtain various formulae and estimates on its kernel, eigenfunctions, and spectrum. The interest in this particular choice is twofold. One is that in order to further develop the more general theory it is important to have cases of reference with as detailed information as possible. The results we obtain below are indeed more refined than the general methods using either pseudo-differential calculus or functional integration provide, and we view this paper as complementary to the more general results in [7, 13]. A second motivation is that there is much controversy in the physics literature (see, for instance, [12, 5]) about claimed solutions of fractional Schrödinger equations. Due to non-locality of these operators such equations are more delicate than usual Schrödinger equations and an appropriate rigorous mathematical treatment is necessary. The operator we consider describes the massless (semi-)relativistic quantum harmonic oscillator studied in physics.

Our main results are as follows. First we derive a functional integral representation which allows to define  $H$  as a self-adjoint operator. From the results of [9] it follows that the first eigenfunction (ground state)  $\varphi_1$  of  $H$  is bounded both from below and above by  $x^{-4}$  with suitable prefactors, moreover, for all other eigenfunctions  $|\varphi_n(x)| \leq \text{const } \varphi_1(x)$  holds. Due to the special choice of the potential, by using special functions we improve this result to a detailed asymptotic expansion of eigenfunctions, in particular, tighten the order of magnitude on the bounds (Theorem 3.10 below). Secondly, in [11] it was proven that the eigenfunctions are uniformly bounded for the case of the Cauchy process run in an interval only. We prove uniform boundedness of all eigenfunctions for  $H$  on  $\mathbb{R}$  (Theorem 3.13), and show that the set of zeroes of each eigenfunction is finite (Corollary 3.12). Also, we discuss the shape of the ground state (Theorem 3.14). On the eigenvalues our main results are the precise asymptotic

expansions in Corollary 3.6 resulting from Theorem 3.5, in which we prove that the eigenvalues are simple. We also obtain a spectral gap estimate (Corollary 3.9), and derive the trace asymptotics in Theorem 3.8. We give a heat kernel estimate in Theorem 3.13.

The plan of the paper is the following. In Section 2 we derive a Feynman-Kac-type formula for a class of operators covering our case and then particularize to our chosen operator. We use the functional representation to define the operator as a self-adjoint operator. In Section 3 we show that the Fourier transform of the eigenfunctions satisfy the Airy equation under appropriate boundary conditions. This allows us to identify the spectrum of the fractional harmonic oscillator operator and derive some asymptotic formulae. Furthermore, here we present the main results as discussed above. In a short Section 4 we provide an appendix of the used facts on Airy functions.

## 2 Functional integral representation

Recall that the linear operator with domain  $H^\alpha(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : |k|^\alpha \hat{f} \in L^2(\mathbb{R}^d)\}$ ,  $0 < \alpha < 2$ ,  $d \geq 1$ , defined by  $(-\Delta)^{\alpha/2} f(k) = |k|^\alpha \hat{f}(k)$ , is the *fractional Laplacian* of order  $\alpha$ . It is essentially self-adjoint on  $C_0^\infty(\mathbb{R}^d)$ , and its spectrum is  $\text{Spec}((-\Delta)^{\alpha/2}) = \text{Spec}_{\text{ess}}((-\Delta)^{\alpha/2}) = [0, \infty)$ .

Let  $V : \mathbb{R}^d \rightarrow \mathbb{R}$  be a bounded Borel measurable function. We call  $(-\Delta)^{\alpha/2} + V$ ,  $0 < \alpha < 2$ , a *fractional Schrödinger operator* with potential  $V$ , where  $V$  acts as a multiplication operator. Since  $V$  is a bounded function, the operator  $(-\Delta)^{\alpha/2} + V$  is self-adjoint on  $\text{Dom}((-\Delta)^{\alpha/2})$  defined as a sum of two self-adjoint operators. Therefore  $\text{Spec}((-\Delta)^{\alpha/2} + V) \subset [0, \infty)$ .

Let  $(\Omega_X, \mathcal{F}_X, P_X)$  be a probability space and  $(X_t)_{t \geq 0}$  a real valued symmetric  $\alpha$ -stable process on it, with  $0 < \alpha < 2$ .  $(X_t)_{t \geq 0}$  is a non-Gaussian Lévy process, in particular it has independent and stationary increments. We use the notations  $P^x$  and  $\mathbb{E}^x$ , respectively, for the distribution and the expected value of the process starting in  $x \in \mathbb{R}$  at time  $t = 0$ ; for simplicity we do not indicate the measure in subscript (while we do when have any other measure or process). The characteristic function of  $(X_t)_{t \geq 0}$  is given by

$$\mathbb{E}^0[e^{i\xi X_t}] = e^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}, t > 0. \quad (2.1)$$

As a Lévy process,  $(X_t)_{t \geq 0}$  has a version with paths in  $D([0, \infty); \mathbb{R}^d)$ , the space of càdlàg functions (i.e., right continuous functions with existing left limits).

Recall that a subordinator  $(S_t)_{t \geq 0}$  on a given probability space  $(\Omega_S, \mathcal{F}_S, P_S)$  is an almost surely non-decreasing  $[0, \infty)$ -valued Lévy process starting at 0. An example is the  $(\alpha/2)$ -stable subordinator  $(S_t)_{t \geq 0}$  uniquely determined by its Laplace transform

$$\mathbb{E}_{P_S}^0[e^{-\lambda S_t}] = e^{-t\lambda^{\alpha/2}}, \quad t \geq 0, \lambda \geq 0. \quad (2.2)$$

Consider standard Brownian motion  $(B_t)_{t \geq 0}$  on a given probability space  $(\Omega_W, \mathcal{F}_W, P_W)$ , where  $P_W$  is Wiener measure. Clearly,

$$\mathbb{E}_{P_W}^0[e^{i\xi B_t}] = e^{-t|\xi|^2}, \quad \xi \in \mathbb{R}, t > 0. \quad (2.3)$$

It is a standard fact that any symmetric  $\alpha$ -stable process  $(X_t)_{t \geq 0}$  can be obtained as a random time change of Brownian motion where this random time process is an  $(\alpha/2)$ -stable subordinator  $(S_t)_{t \geq 0}$ . It is convenient to consider the processes  $(B_t)_{t \geq 0}$  and  $(S_t)_{t \geq 0}$  on two different probability spaces  $(\Omega_W, \mathcal{F}_W, P_W)$  and  $(\Omega_S, \mathcal{F}_S, P_S)$ . Then the process  $(X_t)_{t \geq 0}$  can be obtained in terms of subordinate Brownian motion with respect to the  $(\alpha/2)$ -stable subordinator:

$$X_t : \Omega_{P_W} \times \Omega_{P_S} \ni (\omega, \tau) \mapsto B_{S_t(\tau)}(\omega) := X_t(\omega, \tau).$$

This can also be seen by the composition of the characteristic exponent (2.3) with the Laplace exponent (2.2) which gives (2.1). Furthermore,  $P$  can then be identified as the image measure of this process on  $D([0, \infty); \mathbb{R}^d)$  such that

$$P^x(X_t \in A) = (P_W^x \times P_S^0)(B_{S_t} \in A)$$

holds for all Borel sets  $A \subset \mathbb{R}^d$ .

In what follows we take the case  $d = 1$ ,  $\alpha = 1$ , i.e., the operators  $\sqrt{-\frac{d^2}{dx^2}}$ ,  $V(x) = x^2$  so that we will consider the fractional Schrödinger operator

$$H := \left( -\frac{d^2}{dx^2} \right)^{1/2} + x^2. \quad (2.4)$$

The symmetric 1-stable process  $(X_t)_{t \geq 0}$  is also known as *Cauchy process* whose one-dimensional distributions are given explicitly by

$$P^x(X_t \in dy) = \frac{1}{\pi} \frac{t}{t^2 + (x - y)^2} dy, \quad x \in \mathbb{R}.$$

Our main concern in this paper is to study the spectral properties of  $H$  by using functional integration methods.

First we show how  $H$  relates with the Cauchy process. We have the following Feynman-Kac-type formula, which we state in  $d$  dimensions and a class of  $V$  containing our special case (see [7, 10] for more general pseudo-differential operators).

**Theorem 2.1.** *Let  $V \in L^\infty(\mathbb{R}^d)$  and  $(X_t)_{t \geq 0}$  be a  $d$ -dimensional Cauchy process. We have*

$$(f, e^{-t(\sqrt{-\Delta} + V)}g) = \int_{\mathbb{R}^d} dx \mathbb{E}^x \left[ \overline{f(X_0)} g(X_t) e^{-\int_0^t V(X_s) ds} \right]. \quad (2.5)$$

*Proof.* We divide the proof into four steps.

(Step 1) Suppose  $V \equiv 0$ . Our first claim is

$$(f, e^{-t(\sqrt{-\Delta})}g) = \int_{\mathbb{R}^d} dx \mathbb{E}^x \left[ \overline{f(X_0)} g(X_t) \right]. \quad (2.6)$$

We regard the process  $(X_t)_{t \geq 0}$  as the composition of Brownian motion  $(B_t)_{t \geq 0}$  and the 1/2-stable subordinator  $(S_t)_{t \geq 0}$  as explained above. Let  $E_\lambda$  denote the spectral projection of the self-adjoint operator  $-\Delta \geq 0$ . Then by using (2.2) and the usual Feynman-Kac formula for  $e^{t\Delta}$  we have

$$\begin{aligned} (f, e^{-t\sqrt{-\Delta}}g) &= \int_0^\infty e^{-t\sqrt{\lambda}} d(f, E_\lambda g) = \int_0^\infty \mathbb{E}_{P_S}^0 \left[ e^{-\lambda S_t} \right] d(f, E_\lambda g) \\ &= \mathbb{E}_{P_S}^0 \left[ \int_0^\infty e^{-S_t \lambda} d(f, E_\lambda g) \right] = \mathbb{E}_{P_S}^0 \left[ (f, e^{-S_t(-\Delta)}g) \right] \\ &= \mathbb{E}_{P_S}^0 \left[ \int_{\mathbb{R}^d} dx \mathbb{E}_{P_W}^x \left[ \overline{f(B_0)} g(B_{S_t}) \right] \right] = \int_{\mathbb{R}^d} dx \mathbb{E}^x \left[ \overline{f(X_0)} g(X_t) \right], \end{aligned}$$

thus (2.6) follows.

(Step 2) Let  $0 = t_0 < t_1 < \dots < t_n$ ,  $f_0, f_n \in L^2(\mathbb{R}^d)$  and assume that  $f_j \in L^\infty(\mathbb{R}^d)$ , for  $j = 1, 2, \dots, n-1$ . We claim that

$$\left( f_0, \prod_{j=1}^n e^{-(t_j - t_{j-1})\sqrt{-\Delta}} f_j \right) = \int_{\mathbb{R}^d} dx \mathbb{E}^x \left[ \overline{f(X_0)} \prod_{j=1}^n f_j(X_{t_j}) \right]. \quad (2.7)$$

For simplifying the notation put  $s_j = t_j - t_{j-1}$ , for any  $j = 1, \dots, n$  and

$$g_i = f_i \left( \prod_{j=i+1}^n e^{-s_j \sqrt{-\Delta}} f_j \right), \quad j = 1, \dots, n-1, \quad g_n = f_n.$$

Notice that  $g_j = f_j e^{-s_{j+1} \sqrt{-\Delta}} g_{j+1}$ . By (2.6) the left hand side of (2.7) can be represented as

$$\int_{\mathbb{R}^d} dx \mathbb{E}^x \left[ \overline{f(X_0)} g_1(X_{s_1}) \right] = \int_{\mathbb{R}^d} dx \overline{f(x)} \mathbb{E}^x [g_1(X_{s_1})].$$

Using (2.6) again, we obtain

$$\begin{aligned} \mathbb{E}^x [g_j(X_{s_j})] &= \int_{\mathbb{R}^d} p(s_j, y-x) g_j(y) dy = \int_{\mathbb{R}^d} p(s_j, y-x) f_j(y) e^{-s_{j+1} \sqrt{-\Delta}} g_{j+1}(y) dy \\ &= \int_{\mathbb{R}^d} \mathbb{E}^y [p(s_j, X_0-x) f_j(X_0) g_{j+1}(X_{s_{j+1}})] dy \\ &= \int_{\mathbb{R}^d} p(s_j, y-x) f_j(y) \mathbb{E}^y [g_{j+1}(X_{s_{j+1}})] dy = \mathbb{E}^x [f_j(X_{s_j}) \mathbb{E}^{X_{s_j}} [g_{j+1}(X_{s_{j+1}})]] , \end{aligned}$$

for  $j = 1, \dots, n-1$ . The above equalities yield

$$\begin{aligned} \left( f_0, \prod_{j=1}^n e^{-s_j \sqrt{-\Delta}} f_j \right) &= \int_{\mathbb{R}^d} dx \mathbb{E}^x \left[ \overline{f(X_0)} f_1(X_{s_1}) \times \right. \\ &\quad \left. \times \mathbb{E}^{X_{s_1}} [f_2(X_{s_2}) \mathbb{E}^{X_{s_2}} [f_3(X_{s_3}) \mathbb{E}^{X_{s_3}} [\dots \mathbb{E}^{X_{s_{n-1}} [f_n(X_{s_n})] \dots]}]] \right], \end{aligned}$$

and (2.7) follows by the Markov property of  $(X_t)_{t \geq 0}$ .

(Step 3) Let now  $0 \neq V \in C_b(\mathbb{R}^d)$ . We show (2.5) for such  $V$ . Since  $\sqrt{-\Delta}$  is self-adjoint, the Trotter product formula holds:

$$\left( f, e^{-t(\sqrt{-\Delta}+V)} g \right) = \lim_{n \rightarrow \infty} \left( f, \left( e^{-(t/n)\sqrt{-\Delta}} e^{-(t/n)V} \right)^n g \right).$$

Combined with (Step 2) it yields

$$\left( f, e^{-t(\sqrt{-\Delta}+V)} g \right) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} dx \mathbb{E}^x \left[ \overline{f(X_0)} g(X_t) e^{-\sum_{j=1}^n (t/n)V(X_{t_j/n})} \right].$$

Since each càdlàg path  $s \mapsto \omega(s) = X_s(\omega)$  is continuous in  $s \in [0, t]$  except for at most finite points, we have  $\sum_{j=1}^n (t/n)V(X_{t_j/n}) \rightarrow \int_0^t V(X_s) ds$  as  $n \rightarrow \infty$  in the sense of Riemann integral. Thus (2.5) follows for  $V \in C_b(\mathbb{R}^d)$ .

(Step 4) We make use of the argument in [16, Theorem 6.2] to complete the proof. Suppose that  $V \in L^\infty(\mathbb{R}^d)$  and let  $V_n = \phi(x/n)(V * h_n)$ , where  $h_n = n^d \phi(nx)$  with  $\phi \in C_0^\infty(\mathbb{R}^d)$  such that  $0 \leq \phi \leq 1$ ,  $\int \phi(x) dx = 1$  and  $\phi(0) = 1$ . Then  $V_n \rightarrow V$  almost everywhere and  $V_n$  are bounded and continuous. Let  $\mathcal{N}$  denote the set of all  $x$  such that  $V_n(x)$  does not converge to  $V(x)$ . Then the measure of  $\{t \in [0, \infty) : X_t(\omega) \in \mathcal{N}\}$  is zero  $P^x$ -almost surely and  $\int_0^t V_n(X_s) ds \rightarrow \int_0^t V(X_s) ds$  as  $n \rightarrow \infty$   $P^x$ -a.s. Thus

$$\int_{\mathbb{R}^d} dx \mathbb{E}^x \left[ \overline{f(X_0)} g(X_t) e^{-\int_0^t V_n(X_s) ds} \right] \rightarrow \int_{\mathbb{R}^d} dx \mathbb{E}^x \left[ \overline{f(X_0)} g(X_t) e^{-\int_0^t V(X_s) ds} \right]$$

as  $n \rightarrow \infty$ . On the other hand,  $e^{-t(\sqrt{-\Delta}+V_n)} \rightarrow e^{-t(\sqrt{-\Delta}+V)}$  in strong sense as  $n \rightarrow \infty$ , since  $\sqrt{-\Delta} + V_n \rightarrow \sqrt{-\Delta} + V$  on the domain  $\text{Dom}(\sqrt{-\Delta})$ .  $\square$

We can use Theorem 2.1 to define  $\sqrt{-\Delta}+V$  as a self-adjoint operator. We define the *Feynman-Kac semigroup*

$$(T_t f)(x) = \mathbb{E}^x \left[ e^{-\int_0^t V(X_s) ds} f(X_t) \right], \quad f \in L^2(\mathbb{R}^d), \quad x \in \mathbb{R}^d.$$

**Theorem 2.2.** *Let  $V \in L^1_{\text{loc}}(\mathbb{R}^d)$  such that  $V \geq 0$ . Then  $\{T_t : t \geq 0\}$  is a strongly continuous symmetric semigroup. In particular, there exists a self-adjoint operator  $K$  bounded from below such that  $e^{-tK} = T_t$ .*

$K$  can be identified as the self-adjoint operator  $\sqrt{-\Delta} + V$  for  $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^d)$ .

*Proof.* Since  $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^d)$  there exist  $C_V^{(0)}, C_V^{(1)} > 0$  such that

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}^x [e^{-\int_0^t V(X_s) ds}] \leq e^{C_V^{(0)} + C_V^{(1)} t}. \quad (2.8)$$

Therefore we have

$$\begin{aligned} \|T_t f\|^2 &\leq \int_{\mathbb{R}^d} dx \mathbb{E}^x \left[ e^{-2\int_0^t V(X_s) ds} |f(X_t)|^2 \right] \\ &\leq C_t \int_{\mathbb{R}^d} dx \mathbb{E}^x [|f(X_t)|^2] \\ &= C_t \|e^{(t/2)\Delta} f\|^2 \leq C_t \|f\|^2, \end{aligned}$$

with some  $C_t > 0$ . Thus  $T_t$  is a bounded operator from  $L^2(\mathbb{R}^d)$  to  $L^2(\mathbb{R}^d)$ . Similarly as in Step 2 of the proof of Theorem 2.1 it is seen that the semigroup property  $T_t T_s = T_{t+s}$  holds for  $t, s \geq 0$ .

To obtain strong continuity of  $T_t$  in  $t$  it suffices to show weak continuity. Let  $f, g \in C_0^\infty(\mathbb{R}^d)$ . Then we have

$$(f, T_t g) = \int_{\mathbb{R}^d} dx \mathbb{E}_{P_W \times P_S}^{x,0} \left[ \overline{f(B_0)} g(B_{S_t}) e^{-\int_0^t V(B_{S_r}) dr} \right].$$

Since  $S_t(\tau) \rightarrow 0$  as  $t \rightarrow 0$  for each  $\tau \in \Omega_{P_S}$ , dominated convergence gives  $(f, T_t g) \rightarrow (f, g)$ .

Finally, we check the symmetry property  $T_t^* = T_t$ . Let  $\tilde{B}_s = \tilde{B}_s(\omega, \tau) = B_{S_t(\tau)-s}(\omega) - B_{S_t(\tau)}(\omega)$ . For every  $\tau \in \Omega_{P_S}$  we have then  $\tilde{B}_s \stackrel{d}{=} B_s$  with respect to  $P_W^x$  ( $\stackrel{d}{=}$  denotes that the random variables are identically distributed). Hence

$$\begin{aligned} (f, T_t g) &= \int_{\mathbb{R}^d} dx \overline{f(x)} \mathbb{E}_{P_W \times P_S}^{x,0} \left[ e^{-\int_0^t V(\tilde{B}_{S_r}) dr} g(\tilde{B}_{S_t}) \right] \\ &= \mathbb{E}_{P_W \times P_S}^{0,0} \left[ \int_{\mathbb{R}^d} dx \overline{f(x)} e^{-\int_0^t V(x + \tilde{B}_{S_r}) dr} g(x + \tilde{B}_{S_t}) \right] \\ &= \mathbb{E}_{P_W \times P_S}^{0,0} \left[ \int_{\mathbb{R}^d} dx \overline{f(x - \tilde{B}_{S_t})} e^{-\int_0^t V(x + \tilde{B}_{S_r} - \tilde{B}_{S_t}) dr} g(x) \right]. \end{aligned}$$

In the second equality we changed the variable  $x$  to  $x - \tilde{B}_{S_t}$ . Since  $\tilde{B}_{S_t} \stackrel{d}{=} -B_{S_t}$  and  $\tilde{B}_{S_r} - \tilde{B}_{S_t} \stackrel{d}{=} B_{S_t - S_r}$ , we have

$$(f, T_t g) = \int_{\mathbb{R}^d} dx \mathbb{E}_{P_W \times P_S}^{x,0} \left[ \overline{f(B_{S_t})} e^{-\int_0^t V(B_{S_t - S_r}) dr} g(x) \right].$$

Moreover, as  $S_t - S_r \stackrel{d}{=} S_{t-r}$  for  $0 \leq r \leq t$ , we obtain

$$\begin{aligned} (f, T_t g) &= \int_{\mathbb{R}^d} dx \mathbb{E}_{P_W \times P_S}^{x,0} \left[ \overline{f(B_{S_t})} e^{-\int_0^t V(B_{S_t - r}) dr} g(x) \right] \\ &= \int_{\mathbb{R}^d} dx \mathbb{E}_{P_W \times P_S}^{x,0} \left[ \overline{f(B_{S_t})} e^{-\int_0^t V(B_{S_r}) dr} \right] g(x) = (T_t f, g). \end{aligned}$$

The existence of a self-adjoint operator  $K$  bounded from below such that  $T_t = e^{-tK}$  follows now by the Hille-Yoshida theorem. This completes the proof.  $\square$

### 3 Eigenvalues and eigenfunctions of $H$

#### 3.1 Basic regularity properties

From now on we consider  $H$  defined by (2.4) and the related Feynman-Kac semigroup.  $\{T_t : t \geq 0\}$  is given by an integral kernel, i.e., there exists  $u(t, x, y)$  such that

$$(T_t f)(x) = \int_{\mathbb{R}} u(t, x, y) f(y) dy, \quad x \in \mathbb{R}, f \in L^2(\mathbb{R}).$$

**Lemma 3.1.** *For every  $t > 0$ , the operators  $T_t$  are compact.*

*Proof.* [9], Lemma 1. □

**Lemma 3.2.** *Let  $u(t, x, y)$  be the integral kernel of the Feynman-Kac semigroup  $\{T_t : t \geq 0\}$ . The following properties hold:*

1. *for every  $t > 0$  the function  $u(t, \cdot, \cdot)$  is continuous, strictly positive, and bounded on  $\mathbb{R} \times \mathbb{R}$*
2. *the semigroup is intrinsically ultracontractive, i.e., there exists  $C(t) > 0$  such that  $u(x, y, t) \leq C(t) \varphi_1(x) \varphi_1(y)$ , for all  $t > 0$  and  $x, y \in \mathbb{R}$ , where  $\varphi_1$  is the first eigenfunction of  $\sqrt{-\frac{d^2}{dx^2} + x^2}$ .*

*Proof.* [9]. □

By Lemma 3.1 above the spectrum of  $H$  is purely discrete and there exists an orthonormal basis in  $L^2(\mathbb{R})$  consisting of eigenfunctions  $\varphi_n$  such that  $T_t \varphi_n = e^{-\lambda_n t} \varphi_n$ , where  $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \rightarrow \infty$  are the eigenvalues. Using the relation between the semigroup  $\{T_t : t \geq 0\}$  and its generator  $H$  we get in strong sense

$$H \varphi_n(x) = \lim_{t \downarrow 0} \frac{T_t \varphi_n(x) - \varphi_n(x)}{t} = \lim_{t \downarrow 0} \frac{e^{-\lambda_n t} - 1}{t} \varphi_n(x) = -\lambda_n \varphi_n(x), \quad x \in \mathbb{R},$$

which means that the functions  $(\varphi_n)_{n \in \mathbb{N}}$  are also eigenfunctions of the Schrödinger operator  $H$  and

$$\sqrt{-\frac{d^2}{dx^2}} \varphi_n(x) + x^2 \varphi_n(x) = \lambda_n \varphi_n(x), \quad x \in \mathbb{R}. \quad (3.1)$$

We conclude this section by discussing some basic regularity properties of the eigenfunctions of  $H$ .

**Lemma 3.3.** *For every  $n = 1, 2, \dots$  we have  $\varphi_n \in L^1(\mathbb{R})$ .*

*Proof.* For every  $x \in \mathbb{R}$  define

$$f(x) = \mathbb{E}^x \left[ e^{-\int_0^1 X_s^2 ds} \right]$$

and denote by  $\tau := \inf\{s > 0 : X_s \leq -1\}$  the first exit time of the process  $(X_t)_{t \geq 0}$  from the half-line  $(-1, \infty)$ . Then, for every  $x \in \mathbb{R}$  such that  $x > 2$  we have

$$\begin{aligned} f(x) &= \mathbb{E}^x \left[ e^{-\int_0^1 X_s^2 ds} \right] = \mathbb{E}^0 \left[ e^{-\int_0^1 (X_s + x)^2 ds} \right] \\ &\leq \mathbb{E}^0 \left[ e^{-\int_0^{1 \wedge \tau} (X_s + x)^2 ds} \right] \\ &\leq \mathbb{E}^0 \left[ e^{-(x-1)^2 \int_0^{1 \wedge \tau} ds} \right] = \mathbb{E}^0 \left[ e^{-(x-1)^2 (1 \wedge \tau)} \right] \\ &\leq e^{-(x-1)^2} + \int_1^\infty e^{-(x-1)^2 z} g(z) dz, \end{aligned}$$

where  $g(z)$  is a density function of the random variable  $\tau$ . The explicit formula for  $g$  was derived by Darling [4] (compare also [3, 11]). There exists a constant  $c_1 > 0$  such that for every  $x > 2$  we have  $g(x) < c_1 x^2$  ([3], p. 286). Thus

$$f(x) \leq e^{-(x-1)^2} + c_1 \int_1^\infty e^{-(x-1)^2 z} z^2 dz \leq e^{-(x-1)^2} \left( 1 + c_1 \int_0^\infty e^{-(x-1)^2 u} (u+1)^2 du \right) \leq c_2 e^{-(x-1)^2}$$

whenever  $x > 2$ . The same argument for  $x < -2$  shows that there is a constant  $c_3$  such that for every  $|x| > 2$

$$f(x) \leq c_3 e^{-(|x|-1)^2}.$$

Finally, we get

$$\begin{aligned} \int_{\mathbb{R}} |\varphi_n(y)| dy &= e^{\lambda_n} \int_{\mathbb{R}} |T_1 \varphi_n(y)| dy \leq e^{\lambda_n} \int_{\mathbb{R}} \int_{\mathbb{R}} u(1, x, y) |\varphi_n(x)| dx dy \\ &= e^{\lambda_n} \int_{\mathbb{R}} f(x) |\varphi_n(x)| dx = e^{\lambda_n} \left( \int_{|x| \leq 2} + \int_{|x| > 2} \right) f(x) |\varphi_n(x)| dx \\ &\leq e^{\lambda_n} \left( \int_{-2}^2 |\varphi_n(x)| dx + c_3 \int_{|x| > 2} e^{-(|x|-1)^2} |\varphi_n(x)| dx \right) < \infty, \end{aligned}$$

where the last inequality is a consequence of the fact that  $\varphi_n \in L^2(\mathbb{R})$ . □

In fact, a stronger property is true, which we will need below.

**Lemma 3.4.** *We have that  $x^2 \varphi_n \in L^1(\mathbb{R})$ , for all  $n = 1, 2, \dots$*

*Proof.* For  $V(x) = x^2$  the estimate

$$|\varphi_1(x)| \leq \frac{C}{V(x)|x|^{d+\alpha}}$$

holds for the first eigenfunction (ground state)  $\varphi_1(x)$ , see [9], Theorem 1. Since the Feynman-Kac semigroup is intrinsically ultracontractive by Lemma 3.2 above,

$$|\varphi_n(x)| \leq \frac{c_n}{x^4}, \quad |x| > 1 \tag{3.2}$$

follows. Then the proof of the claim is straightforward. □

## 3.2 Eigenvalues

We will determine the functions  $\varphi_n$  and corresponding eigenvalues  $\lambda_n$  starting from the relation (3.1). Denote the Fourier transform of eigenfunctions by

$$y_{\lambda_n}(x) := \widehat{\varphi}_n(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixz} \varphi_n(z) dz.$$

Note that  $y_{\lambda_n}$  is well-defined since  $\varphi_n \in L^1(\mathbb{R})$  by Lemma 3.3. Moreover, Lemma 3.4 implies that  $y_{\lambda_n} \in C^2(\mathbb{R})$ ,  $n = 1, 2, \dots$  By performing Fourier transform in (3.1)

$$-y_{\lambda_n}''(x) + |x| y_{\lambda_n}(x) = \lambda_n y_{\lambda_n}(x) \tag{3.3}$$

is obtained. We are looking for  $y_\lambda \in C^2(\mathbb{R}) \cap L^1(\mathbb{R})$  satisfying

$$-y_\lambda''(x) + |x| y_\lambda(x) = \lambda y_\lambda(x), \quad \lambda > 0. \tag{3.4}$$

Notice that if a function  $y_\lambda(x)$  is a solution of (3.4), then the function  $y_\lambda(-x)$  is also a solution. Hence it suffices to consider equation (3.4) only for  $x > 0$  and construct even and odd solutions on the whole real line.

For  $x > 0$  equation (3.4) takes the form

$$y_\lambda''(x) - (x - \lambda)y_\lambda(x) = 0.$$

On substituting  $z = x - \lambda$ ,  $f_\lambda(z) = y_\lambda(x)$  the equation reduces to the Airy differential equation (4.1) below

$$f_\lambda''(z) - zf_\lambda(z) = 0.$$

(For a discussion of Airy functions see the Appendix.) The general solution of the above equation is thus obtained as

$$y_\lambda(x) = c_1 \text{Ai}(x - \lambda) + c_2 \text{Bi}(x - \lambda).$$

The facts that  $y \in L^1(\mathbb{R})$  and the function  $\text{Bi}(z)$  tends to infinity when  $z \rightarrow \infty$  (see (4.3)) imply  $c_2 = 0$ . Without loss of generality we can assume that  $c_1 = 1$ .

Finally, to obtain an even function on the whole real line we put

$$y_\lambda(x) = \text{Ai}(|x| - \lambda), \quad x \in \mathbb{R}. \quad (3.5)$$

We furthermore require that the right derivative at 0 is zero

$$\lim_{x \rightarrow 0^+} y_\lambda'(x) = 0$$

which is equivalent to

$$\text{Ai}'(-\lambda) = 0. \quad (3.6)$$

For odd functions we have

$$y_\lambda(x) = \text{sgn}(x)\text{Ai}(|x| - \lambda), \quad x \in \mathbb{R} \quad (3.7)$$

and we require

$$\lim_{x \rightarrow 0^+} y_\lambda(x) = 0$$

or equivalently

$$\text{Ai}(-\lambda) = 0. \quad (3.8)$$

Conditions (3.6) and (3.8) together with equation (3.4) imply that the functions  $y_\lambda$  defined on the real line by (3.5) and (3.7) belong to  $C^2(\mathbb{R})$ .

By the Parseval equality and the fact that  $(x\text{Ai}^2(x) - (\text{Ai}(x))^2)' = \text{Ai}(x)$ , which is an easy consequence of the Airy equation (4.1), we get

$$\int_{-\infty}^{\infty} \varphi_n^2(x) dx = \int_{-\infty}^{\infty} (\widehat{\varphi}_n)^2(x) dx = 2 \int_0^{\infty} \text{Ai}(x + \lambda_n)^2 dx = 2 ((\text{Ai}'(-\lambda_n))^2 + \lambda_n \text{Ai}^2(-\lambda_n)).$$

We have thus proved the following result.

**Theorem 3.5.** *The eigenvalues for the problem (3.1) are given by*

$$\begin{aligned}\lambda_{2k-1} &= -a'_k & k = 1, 2, \dots, \\ \lambda_{2k} &= -a_k, & k = 1, 2, \dots,\end{aligned}$$

where  $a_k$  and  $a'_k$  denote the zeroes of the functions  $\text{Ai}$  and  $\text{Ai}'$  in decreasing order. They are all simple, the eigenfunctions  $\varphi_{2k-1}(x)$  are even and  $\varphi_{2k}(x)$  are odd. Furthermore, the Fourier transforms of the  $L^2$ -normalized eigenfunctions are given by

$$\widehat{\varphi}_n(x) = \begin{cases} \frac{\text{Ai}(|x| - \lambda_n)}{\sqrt{2\lambda_n}\text{Ai}(-\lambda_n)} & n = 1, 3, 5, \dots \\ \frac{\text{sgn}(x)\text{Ai}(|x| - \lambda_n)}{\sqrt{2}\text{Ai}'(-\lambda_n)} & n = 2, 4, 6, \dots \end{cases}, \quad x \in \mathbb{R}.$$

Making use of the asymptotic expansions and estimates for the zeroes of the Airy function and its derivative [14, 6] yields

**Corollary 3.6.** *We have*

$$\begin{aligned}\lambda_{2k-1} &\sim g\left(\frac{3}{8}\pi(4k-3)\right), & k \rightarrow \infty, \\ \lambda_{2k} &\sim f\left(\frac{3}{8}\pi(4k-1)\right), & k \rightarrow \infty.\end{aligned}$$

where

$$g(t) = t^{2/3} \left(1 - \frac{7}{48}t^{-2} + \frac{35}{288}t^{-4} - \frac{181223}{207360}t^{-6} + \frac{18683371}{1244160}t^{-8} - \frac{91145884361}{191102976}t^{-10}\right), \quad (3.9)$$

$$f(t) = t^{2/3} \left(1 + \frac{5}{48}t^{-2} - \frac{5}{36}t^{-4} + \frac{77125}{82944}t^{-6} - \frac{108056875}{6967296}t^{-8} + \frac{162375596875}{334430208}t^{-10}\right). \quad (3.10)$$

Moreover, we have

$$\begin{aligned}\lambda_{2k-1} &\leq \left(\frac{3\pi}{8}(4k-1)\right)^{2/3}, & k = 1, 2, \dots, \\ \left(\frac{3\pi}{8}(4k-1)\right)^{2/3} &\leq \lambda_{2k} \leq \left(\frac{3\pi}{8}(4k-1)\right)^{2/3} \left(1 + \frac{3}{2} \arctan\left(\frac{5}{18\pi(4k-1)}\right)\right), & k = 1, 2, \dots\end{aligned}$$

**Remark 3.7.** A numerical calculation for the first few eigenvalues gives

$$\begin{aligned}\lambda_1 &\cong 1.01879297164747 \\ \lambda_2 &\cong 2.33810741045976 \\ \lambda_3 &\cong 3.24819758217983 \\ \lambda_4 &\cong 4.08794944413097 \\ \lambda_5 &\cong 4.82009921117874 \\ \lambda_6 &\cong 5.52055982809555.\end{aligned}$$

Using the above asymptotic formulae we can derive a result on the asymptotic behaviour of the trace of the semigroup at zero.

**Theorem 3.8.** *We have*

$$\lim_{t \rightarrow 0^+} t^{3/2} \sum_{n=1}^{\infty} e^{-\lambda_n t} = \frac{1}{\sqrt{\pi}}.$$

*Proof.* We divide the series into two components

$$F(t) = \sum_{k=1}^{\infty} e^{-\lambda_{2k-1} t} = \sum_{k=1}^{\infty} e^{a'_k t}, \quad G(t) = \sum_{k=1}^{\infty} e^{-\lambda_{2k} t} = \sum_{k=1}^{\infty} e^{a_k t}.$$

By (3.9) we get that for every  $t \in [0, 1]$

$$\sum_{k=1}^{\infty} e^{-(3/2\pi(k-3/4)^{2/3}t)} \leq F(t) \leq e^{-ct} \sum_{k=1}^{\infty} e^{-(3/2\pi(k-3/4)^{2/3}t)}$$

for some constant  $c > 0$ . Moreover, the function  $x \mapsto e^{-(3/2\pi(x-3/4)^{2/3}t)}$  is strictly positive and non-increasing on  $[1, \infty)$ . Thus

$$e^{-ct} \int_1^{\infty} e^{-(3/2\pi(x-3/4)^{2/3}t)} dx \leq F(t) \leq \left( e^{-(3/8\pi)^{2/3}t} + \int_1^{\infty} e^{-(3/2\pi(x-3/4)^{2/3}t)} dx \right).$$

A substitution yields

$$\int_1^{\infty} e^{-(3/2\pi(x-3/4)^{2/3}t)} dx = \frac{1}{t^{3/2}\pi} \int_{(3/8\pi)^{2/3}t}^{\infty} e^{-u} u^{1/2} du$$

and we obtain  $\lim_{t \rightarrow 0^+} t^{3/2} F(t) = \pi^{-1} \Gamma(3/2) = \frac{1}{2\sqrt{\pi}}$ . In the same way we get  $\lim_{t \rightarrow 0^+} t^{3/2} G(t) = \frac{1}{2\sqrt{\pi}}$  and this completes the proof.  $\square$

Using the estimates for the eigenvalues given in Corollary 3.6 we obtain an estimate for the spectral gap.

**Corollary 3.9.** *We have*

$$\lambda_2 - \lambda_1 \geq \left( \frac{3\pi}{8} \right)^{2/3} (3^{2/3} - 1).$$

### 3.3 Eigenfunctions

Next we derive the asymptotic behaviour of the eigenfunctions. We use the notation  $p_n, q_n$  as in (4.4) below.

**Theorem 3.10.** *For every  $k = 1, \dots$  and  $N = 2, 3, \dots$  we have*

$$\varphi_{2k-1}(z) = \sqrt{\frac{2}{-a'_k}} \left( \frac{p_3(a'_k)}{z^4} - \frac{p_5(a'_k)}{z^6} + \dots + (-1)^N \frac{p_{2N-1}(a'_k)}{z^{2N}} \right) + O\left( \frac{1}{z^{2N+2}} \right), \quad \text{as } |z| \rightarrow \infty.$$

*For every  $k = 1, 2, \dots$  and  $N = 2, 3, \dots$  we have*

$$\varphi_{2k}(z) = \sqrt{2} \left( \frac{q_4(a_k)}{z^5} - \frac{q_6(a_k)}{z^7} + \dots + (-1)^N \frac{q_{2N}(a_k)}{z^{2N+1}} \right) + O\left( \frac{1}{z^{2N+3}} \right), \quad \text{as } |z| \rightarrow \infty.$$

*Proof.* For  $k = 1, 2, \dots$ , we have  $\lambda_{2k-1} = -a'_k$ , and

$$\varphi_{2k-1}(z) = \sqrt{\frac{2}{-a'_k}} \frac{1}{\text{Ai}(a'_k)} \int_0^\infty \text{Ai}(u + a'_k) \cos zu \, du. \quad (3.11)$$

Integration by parts ( $2N + 2$  times) together with (4.5), (4.7) and (4.4) give

$$\begin{aligned} \int_0^\infty \text{Ai}(u + a'_k) \cos zu \, du &= \sum_{s=0}^N \frac{\sin zu}{z^{2s+1}} (-1)^s \text{Ai}^{(2s)}(u + a'_k) \Big|_0^\infty + \\ &\quad + \sum_{s=1}^N \frac{\cos zu}{z^{2s}} (-1)^{s-1} \text{Ai}^{(2s-1)}(u + a'_k) \Big|_0^\infty + R_N(z) \\ &= \sum_{s=1}^N \frac{1}{z^{2s}} (-1)^s \text{Ai}^{(2s-1)}(a'_k) + R_N(z) \\ &= \sum_{s=1}^N \frac{p_{2s-1}(a'_k)}{z^{2s}} (-1)^s \text{Ai}(a'_k) + R_N(z), \end{aligned}$$

where

$$\begin{aligned} R_N(z) &= \frac{\cos zu}{z^{2N+2}} \text{Ai}^{(2N+1)}(u + a'_k) \Big|_0^\infty + \frac{1}{z^{2N+2}} \int_0^\infty \text{Ai}^{(2N+2)}(u + a'_k) \cos zu \, du \\ &= -\frac{p_{2N+1}(a'_k) \text{Ai}(a'_k)}{z^{2N+2}} + \frac{1}{z^{2N+2}} \int_0^\infty \text{Ai}^{(2N+2)}(u + a'_k) \cos zu \, du. \end{aligned}$$

Using the asymptotic relations (4.5) and (4.7) together with formula (4.4) we get

$$|R_N(z)| \leq \frac{1}{|z|^{2N+2}} \left( |p_{2N+1}(a'_k) \text{Ai}(a'_k)| + \int_0^\infty |\text{Ai}^{(2N+2)}(u + a'_k)| \, du \right) = \frac{c_{2k-1, N}}{|z|^{2N+2}}.$$

Notice that  $p_1(x) \equiv 0$ , which completes the proof for this case.

For  $k = 1, 2, \dots$  we have

$$\varphi_{2k}(z) = \frac{\sqrt{2}}{\text{Ai}'(a_k)} \int_0^\infty \text{Ai}(u + a_k) \sin zu \, du.$$

Similar arguments give

$$\begin{aligned} \frac{\text{Ai}'(a_k)}{\sqrt{2}} \varphi_{2k}(z) &= \sum_{s=0}^N \frac{\cos zu}{z^{2s+1}} (-1)^{s+1} \text{Ai}^{(2s)}(u + a_k) \Big|_0^\infty + \sum_{s=0}^N \frac{\sin zu}{z^{2s+2}} (-1)^s \text{Ai}^{(2s+1)}(u + a_k) \Big|_0^\infty + R_N(z) \\ &= \sum_{s=0}^N \frac{(-1)^s}{z^{2s+1}} \text{Ai}^{(2s)}(a_k) + R_N(z) \\ &= \sum_{s=1}^N \frac{(-1)^s q_{2s}(a_k)}{z^{2s+1}} \text{Ai}'(a_k) + R_N(z), \end{aligned}$$

where

$$\begin{aligned} R_N(z) &= \frac{\cos zu}{z^{2N+3}} (-1)^{N+2} \text{Ai}^{(2N+2)}(u + a_k) \Big|_0^\infty + \frac{(-1)^{N+1}}{z^{2N+3}} \int_0^\infty \text{Ai}^{(2N+3)}(u + a_k) \cos zu \, du \\ &= \frac{1}{z^{2N+3}} \left( q_{2N+3}(a_k) \text{Ai}'(a_k) + \int_0^\infty \text{Ai}^{(2N+3)}(u + a_k) \cos zu \, du \right). \end{aligned}$$

Thus we get  $|R_N(z)| \leq c_{2k, N} |z|^{-2N-3}$ . Finally, notice that  $q_2(x) \equiv 0$ . This completes the proof.  $\square$

**Theorem 3.11.** *The eigenfunctions  $\varphi_n$  are analytic functions on  $\mathbb{R}$ . Their Maclaurin expansions are given by*

$$\begin{aligned}\varphi_{2k-1}(x) &= \sqrt{\frac{2}{-a'_k}} \frac{1}{\text{Ai}(a'_k)} \sum_{m=0}^{\infty} \frac{w_{2m}(a'_k)(-1)^m}{(2m)!} x^{2m}, \\ \varphi_{2k}(x) &= \frac{\sqrt{2}}{\text{Ai}'(a_k)} \sum_{m=0}^{\infty} \frac{w_{2m+1}(a_k)(-1)^m}{(2m+1)!} x^{2m+1},\end{aligned}$$

where  $k = 1, 2, \dots$  and

$$w_n(x) = \int_0^{\infty} \text{Ai}(u+x)u^n du.$$

*Proof.* For  $k = 1, 2, \dots$  we have

$$\begin{aligned}\varphi_{2k-1}(x) &= \sqrt{\frac{2}{-a'_k}} \frac{1}{\text{Ai}(a'_k)} \int_0^{\infty} \text{Ai}(u+a'_k) \cos x u du \\ &= \sqrt{\frac{2}{-a'_k}} \frac{1}{\text{Ai}(a'_k)} \int_0^{\infty} \text{Ai}(u+a'_k) \sum_{m=0}^{\infty} \frac{(xu)^{2m}}{(2m)!} (-1)^m du.\end{aligned}$$

Moreover, by (4.5) it is seen that there exists a constant  $c_k > 0$  such that  $|\text{Ai}(u+a'_k)| < c_k e^{-\frac{2}{3}u^{3/2}}$  for all  $u > 0$ . Hence

$$\begin{aligned}\sum_{m=0}^{\infty} \int_0^{\infty} \left| \frac{x^{2m} u^{2m}}{(2m)!} (-1)^m \text{Ai}(u+a'_k) \right| du &= \sum_{m=0}^{\infty} \frac{|x|^{2m}}{(2m)!} \int_0^{\infty} |\text{Ai}(u+a'_k)| u^{2m} du \\ &\leq c_k \sum_{m=0}^{\infty} \frac{|x|^{2m}}{(2m)!} \int_0^{\infty} e^{-\frac{2}{3}u^{3/2}} u^{2m} du \\ &= c_k \sum_{m=0}^{\infty} \frac{(|x|^2)^m}{(2m)!} \left(\frac{3}{2}\right)^{\frac{4m-1}{3}} \Gamma\left(\frac{4m+2}{3}\right).\end{aligned}$$

By putting  $d_m = \frac{1}{(2m)!} \left(\frac{3}{2}\right)^{\frac{4m-1}{3}} \Gamma\left(\frac{4m+2}{3}\right)$  and making use of Stirling's formula  $\Gamma(x) \sim \sqrt{2\pi} e^{-x} x^{x-1/2}$ , as  $x \rightarrow \infty$  we obtain

$$\begin{aligned}\frac{d_m}{d_{m+1}} &= (2m+1)(2m+2) \left(\frac{2}{3}\right)^{4/3} \frac{\Gamma\left(\frac{4m+2}{3}\right)}{\Gamma\left(\frac{4m+6}{3}\right)} \\ &\sim (2m+1)(2m+2) \left(\frac{2}{3}\right)^{4/3} e^{4/3} \left(\frac{4m+2}{4m+6}\right)^{\frac{4m+2}{3}-\frac{1}{2}} \left(\frac{3}{4m+6}\right)^{4/3} \xrightarrow{m \rightarrow \infty} \infty.\end{aligned}$$

Thus Fubini's theorem applies to get

$$\varphi_{2k-1}(x) = \sqrt{2} \sum_{m=0}^{\infty} \frac{(-1)^m}{\sqrt{-a'_k} (2m)!} \left( \frac{1}{\text{Ai}(a'_k)} \int_0^{\infty} \text{Ai}(u+a'_k) u^{2m} du \right) x^{2m}.$$

Similar arguments give

$$\varphi_{2k}(x) = \sqrt{2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \left( \frac{1}{\text{Ai}'(a_k)} \int_0^{\infty} \text{Ai}(u+a_k) u^{2m+1} du \right) x^{2m+1}.$$

□

**Corollary 3.12.** *Every eigenfunction  $\varphi_n$  has a finite number of zeroes.*

*Proof.* Using the asymptotic expansions in Theorem 3.10 it is easily seen that for every  $n$  there exists  $A_n > 0$  such that  $\sup_{|x| > A_n} |\varphi_n(x)| > 0$ . This means that all zeroes of  $\varphi_n$  are in  $[-A_n, A_n]$ . Since the function  $\varphi_n$  is analytic, the set of its zeroes is finite.  $\square$

**Theorem 3.13.** *The eigenfunctions  $\varphi_n$  are uniformly bounded.*

*Proof.* We begin with the case of odd eigenfunctions  $\varphi_n$ , where  $n = 2k - 1$ ,  $k = 1, 2, \dots$ ; the proof for the even eigenfunctions is similar. It suffices to consider  $x \geq 0$ . We have

$$\varphi_n(x) = \sqrt{\frac{2}{-a'_k}} \frac{1}{\text{Ai}(a'_k)} \int_0^\infty \text{Ai}(u + a'_k) \cos zu \, du.$$

Using the asymptotic formulae (3.9), (4.8) for  $-a'_k$  and  $\text{Ai}(a'_k)$  respectively we get that

$$\frac{1}{\sqrt{-a'_k} \text{Ai}(a'_k)} = O(k^{-1/6}) \quad (3.12)$$

We have

$$\int_0^\infty \text{Ai}(u + a'_k) \cos x u \, du = \int_{-a'_1}^{-a'_k} \text{Ai}(u) \cos x(u + a'_k) \, du + \int_{a'_1}^\infty \text{Ai}(u) \cos x(u - a'_k) \, du.$$

The Airy function is non-negative on  $[a'_1, \infty]$  and thus the absolute value of the second integral is uniformly bounded by  $\int_{a'_1}^\infty \text{Ai}(u) \, du < \infty$ . The first integral is a sum of two integrals  $I_1(x)$  and  $I_2(x)$  where

$$\begin{aligned} I_1(x) &= \int_{-a'_1}^{-a'_k} \left( \text{Ai}(u) - \frac{\sin\left(\frac{2}{3}u^{3/2} + \frac{\pi}{4}\right)}{u^{1/4}\sqrt{\pi}} \right) \cos x(u + a'_k) \, du \\ I_2(x) &= \frac{1}{\sqrt{\pi}} \int_{-a'_1}^{-a'_k} \sin\left(\frac{2}{3}u^{3/2} + \frac{\pi}{4}\right) \frac{\cos x(u + a'_k)}{u^{1/4}} \, du. \end{aligned}$$

Using the asymptotic expansion for the Airy function ([1] 10.4.60 p.448) we get

$$\text{Ai}(u) - \frac{\sin\left(\frac{2}{3}u^{3/2} + \frac{\pi}{4}\right)}{u^{1/4}\sqrt{\pi}} = O\left(\frac{1}{u^{7/4}}\right),$$

as  $u \rightarrow \infty$ . Thus there is a constant  $c_1 > 0$  such that

$$|I_1(x)| \leq c_1 \int_{-a'_1}^\infty u^{-7/4} \, du.$$

The integral  $2\sqrt{\pi}I_2(x)$  can be rewritten as the sum  $I_3(x) + I_4(x)$ , where

$$\begin{aligned} I_3(x) &= \int_{-a'_1}^{-a'_k} (u^{1/2} + x) \sin\left(\frac{2}{3}u^{3/2} + \frac{\pi}{4} + xu + xa'_k\right) \frac{du}{(u^{1/2} + x)u^{1/4}} \\ I_4(x) &= \int_{-a'_1}^{-a'_k} \sin\left(\frac{2}{3}u^{3/2} + \frac{\pi}{4} - xu - xa'_k\right) \frac{du}{u^{1/4}}. \end{aligned}$$

The term  $I_3$  is uniformly bounded by Lemma 4.1 with

$$f(x, u) = (u^{1/2} + x) \sin\left(\frac{2}{3}u^{3/2} + \frac{\pi}{4} + xu + xa'_k\right)$$

and

$$g(x, u) = (u^{1/2} + x)^{-1} u^{-1/4}.$$

To deal with the term  $I_4$  we need to consider several cases. Let  $(x-1)^2 > -a'_k$  or  $(x+1)^2 < -a'_1$  and rewrite  $I_4$  in the form

$$I_4(x) = \int_{-a'_1}^{-a'_k} (u^{1/2} - x) \sin\left(\frac{2}{3}u^{3/2} + \frac{\pi}{4} - xu - xa'_k\right) \frac{du}{u^{1/4}(u^{1/2} - x)}.$$

An application of Lemma 4.1 with

$$f(x, u) = (u^{1/2} - x) \sin\left(\frac{2}{3}u^{3/2} + \frac{\pi}{4} - xu - xa'_k\right)$$

and

$$g(x, u) = (u^{1/2} - x)^{-1} u^{-1/4}$$

implies that  $I_4$  is uniformly bounded for  $x > \sqrt{-a'_k} + 1$  and  $k = 2, 3, \dots$ . For the case  $-a'_1 < (x-1)^2 \leq -a'_k \leq (x+1)^2$  we have

$$\begin{aligned} I_4(x) &= \int_{-a'_1}^{(x-1)^2} (u^{1/2} - x) \sin\left(\frac{2}{3}u^{3/2} + \frac{\pi}{4} - xu - xa'_k\right) \frac{du}{u^{1/4}(u^{1/2} - x)} \\ &\quad + \int_{(x-1)^2}^{-a'_k} \sin\left(\frac{2}{3}u^{3/2} + \frac{\pi}{4} - xu - xa'_k\right) \frac{du}{u^{1/4}}. \end{aligned}$$

The first integral above is uniformly bounded by the same argument as in the previous case. The absolute value of the second integral is bounded by

$$\int_{(x-1)^2}^{-a'_k} \frac{du}{u^{1/4}} \leq \int_{(x-1)^2}^{(x+1)^2} \frac{du}{u^{1/4}} = \frac{3}{4}((x+1)^{3/4} - (x-1)^{3/4}) \leq c_1 x^{1/4} \leq c_2 k^{1/6}.$$

The last inequality follows from the fact that  $(x-1)^2 \leq -a'_k$  and the asymptotic expansion for  $a'_k$ . In the case  $-a'_1 < (x-1)^2 \leq (x+1)^2 < -a'_k$  we split up  $I_4$  as

$$\begin{aligned} I_4(x) &= \left( \int_{-a'_1}^{(x-1)^2} + \int_{(x+1)^2}^{-a'_k} \right) (u^{1/2} - x) \sin\left(\frac{2}{3}u^{3/2} + \frac{\pi}{4} - xu - xa'_k\right) \frac{du}{u^{1/4}(u^{1/2} - x)} \\ &\quad + \int_{(x-1)^2}^{-a'_k} \sin\left(\frac{2}{3}u^{3/2} + \frac{\pi}{4} - xu - xa'_k\right) \frac{du}{u^{1/4}}. \end{aligned}$$

By Lemma 4.1 the first two integrals are uniformly bounded following by the same argument as before. The last integral is bounded by  $c_2 k^{1/6}$ . Finally, when  $(x-1)^2 < -a'_1 < (x+1)^2 < -a'_k$  we have

$$\begin{aligned} I_4(x) &= \int_{-a'_1}^{(x+1)^2} \sin\left(\frac{2}{3}u^{3/2} + \frac{\pi}{4} - xu - xa'_k\right) \frac{du}{u^{1/4}} \\ &\quad + \int_{(x-1)^2}^{-a'_k} (u^{1/2} - x) \sin\left(\frac{2}{3}u^{3/2} + \frac{\pi}{4} - xu - xa'_k\right) \frac{du}{u^{1/4}(u^{1/2} - x)}. \end{aligned}$$

The absolute value of the first integral is estimated as

$$\int_{-a'_1}^{(x+1)^2} \frac{du}{u^{1/4}} \leq \int_{-a'_1}^{(\sqrt{-a'_1}+2)^2} \frac{du}{u^{1/4}}.$$

The last term can be uniformly bounded by one more application of Lemma 4.1. Hence we obtain

$$\int_0^\infty \text{Ai}(u + a'_k) \cos x u du = O(k^{1/6})$$

uniformly in  $x \geq 0$ . Together with (3.12) this implies that the functions  $\varphi_n$  are uniformly bounded.  $\square$

**Theorem 3.14.** *The ground state  $\varphi_1$  is decreasing on  $(0, \infty)$ . Moreover, there exist  $x_1 > x_0 > 0$  such that  $\varphi_1$  is concave on  $[-x_0, x_0]$  and is convex on  $(-\infty, -x_1]$  and  $[x_1, \infty)$ .*

*Proof.* For  $0 < x < y$ , let  $(R_t^{(1)})_{t \geq 0}$ ,  $(R_t^{(2)})_{t \geq 0}$  be two squared-Bessel processes of dimension 1 and with index  $\nu = -1/2$ , such that  $R_0^{(1)} = x^2$  and  $R_0^{(2)} = y^2$ , and let  $(\eta_t)_{t \geq 0}$  be a  $1/2$ -stable subordinator independent from  $R^{(1)}$  and  $R^{(2)}$ . Using the comparison theorem (see [15], Chapter IX, Theorem 3.7) we get  $R_t^{(1)} \leq R_t^{(2)}$  for all  $t \geq 0$  with probability 1. Thus  $R_{\eta_t}^{(1)} \leq R_{\eta_t}^{(2)}$ , for all  $t \geq 0$ . However, the process  $R_{\eta_t}^{(1)}$  is  $X_t^2$  starting from  $x^2$ , and  $R_{\eta_t}^{(2)}$  is  $X_t^2$  starting from  $y^2$ . Hence

$$\mathbb{E}^x \left[ e^{-\int_0^t X_s^2 ds} \right] \geq \mathbb{E}^y \left[ e^{-\int_0^t X_s^2 ds} \right], \quad 0 < x < y.$$

For every  $x \in \mathbb{R}$  we have

$$\begin{aligned} \mathbb{E}^x \left[ e^{-\int_0^t X_s^2 ds} \right] &= \int_{\mathbb{R}} u(t, x, y) dy \\ &= \int_{\mathbb{R}} \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y) dy \\ &= \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x) \int_{\mathbb{R}} \varphi_n(y) dy. \end{aligned}$$

This implies

$$\varphi_1(x) = \left( \int_{\mathbb{R}} \varphi_n(y) dy \right)^{-1} \lim_{t \rightarrow \infty} e^{\lambda_1 t} \mathbb{E}^x \left[ e^{-\int_0^t X_s^2 ds} \right]$$

and therefore the ground state is non-increasing on  $(0, \infty)$  as a limit of non-increasing functions. In fact,  $\varphi_1$  is strictly decreasing on  $(0, \infty)$ . This easily follows from the monotonicity of  $\varphi_1$  proven above, and the fact that  $\varphi_1$  is analytic on  $\mathbb{R}$ .

Concavity follows from the expression

$$\varphi_1''(0) = -\sqrt{\frac{2}{-a'_1}} \frac{1}{\text{Ai}(a'_1)} \int_0^\infty \text{Ai}(u + a'_1) u^2 du.$$

We have  $\text{Ai}(u + a'_1) u^2 > 0$  on  $(0, \infty)$  thus  $\varphi_1''(0) < 0$ . By continuity of  $\varphi_1''$  at 0 it follows that there exists  $x_0 > 0$  such that  $\varphi_1''(x) < 0$  for all  $x \in [-x_0, x_0]$ . Moreover, we have

$$\varphi_1(x) = -\sqrt{\frac{2}{-a'_1}} \frac{1}{\text{Ai}(a'_1)} \int_0^\infty \text{Ai}(u + a'_1) u^2 \cos x u du.$$

Putting  $f(u) = \text{Ai}(u + a'_1) u^2$  and integrating by parts we obtain

$$\frac{1}{\text{Ai}(a'_1)} \int_0^\infty \text{Ai}(u + a'_1) u^2 \cos x u du = -\frac{f^{(5)}(0)}{x^6} + \frac{1}{x^7} \int_0^\infty f^{(7)}(u) \sin x u du,$$

where  $f^{(5)}(0) = 20\text{Ai}(a'_1)$  and  $f^{(7)}(u) = P(u)\text{Ai}(u + a'_1) + Q(u)\text{Ai}'(u + a'_1)$ , where  $P(u)$  and  $Q(u)$  are polynomials. Applying (4.5) and (4.7) we get

$$\lim_{|x| \rightarrow \infty} x^6 \varphi_1''(x) = 20 \sqrt{\frac{2}{-a_1}} > 0$$

and this implies that there exists  $x_1 > x_0$  such that  $\varphi_1''$  is positive on  $(-\infty, -x_1]$  and  $[x_1, \infty)$ . This completes the proof.  $\square$

**Theorem 3.15.** *The integral kernel  $u(t, x, y)$  is jointly continuous on  $(0, \infty) \times \mathbb{R} \times \mathbb{R}$ . Moreover, there exists a constant  $c > 1$  such that*

$$\frac{1}{c} \frac{e^{a'_1 t}}{(1+x^4)(1+y^4)} \leq u(t, x, y) \leq c \frac{e^{a'_1 t}}{(1+x^4)(1+y^4)}, \quad (3.13)$$

for every  $t > 1$  and  $x, y \in \mathbb{R}$ .

*Proof.* For every  $t_0 > 0$ , using the uniform boundedness of the eigenfunctions  $\varphi_n$  given in Theorem 3.13, we have

$$|u(t, x, y)| \leq \sum_{n=1}^{\infty} e^{-\lambda_n t} |\varphi_n(x)| |\varphi_n(y)| \leq M \sum_{n=1}^{\infty} e^{-\lambda_n t} \leq M \sum_{n=1}^{\infty} e^{-\lambda_n t_0} < \infty, \quad x, y \in \mathbb{R}, t > t_0,$$

for some constant  $M > 0$ . The boundedness of the last series easily follows from the asymptotic expansions for  $\lambda_n$  given in Corollary 3.6. Then the continuity of  $u(t, x, y)$  follows from the continuity of  $\varphi_n$ .

For  $k = 1, 2, \dots$  we have

$$\frac{\sqrt{-a'_k}}{\sqrt{2}} \varphi_{2k-1}(x) = \frac{1}{x^4} \left( 1 + \frac{1}{\text{Ai}(a'_k)} \int_0^{\infty} \text{Ai}^{(4)}(u + a'_k) \cos x u du \right),$$

where  $\text{Ai}^{(4)}(t) = t^2 \text{Ai}(t) + 2\text{Ai}'(t)$ . From the asymptotic expansions ([1] 10.4.60 and 10.4.62) one can easily get that

$$|\text{Ai}^{(4)}(t)| \leq c_1 t^2,$$

with some constant  $c_1$ . Thus, using (4.5) and (4.7), we get

$$\left| \int_0^{\infty} \text{Ai}^{(4)}(u + a'_k) \cos x u du \right| \leq \int_{a'_k}^0 |\text{Ai}^{(4)}(t)| dt + \int_0^{\infty} \text{Ai}^{(4)}(t) dt \leq c_1 \int_{a'_k}^0 t^2 dt + c_2 \leq c_3 (-a'_k)^3$$

Similarly, we have

$$\frac{\text{Ai}'(a_k)}{\sqrt{2}} \varphi_{2k}(x) = \frac{1}{x^4} \int_0^{\infty} \text{Ai}^{(4)}(u + a_k) \sin x u du,$$

where

$$\left| \int_0^{\infty} \text{Ai}^{(4)}(u + a_k) \sin x u du \right| \leq c_3 (-a_k)^3$$

Using the asymptotic formulae (4.8) and (4.9) we obtain that  $|x^4 \varphi_{2k-1}(x)| \leq c_4 (k - 3/4)^{11/6}$  and  $|x^4 \varphi_{2k}(x)| \leq c_4 (k - 1/2)^{11/6}$ . This yields, combined with the asymptotic expansion for  $\varphi_1$ , the estimates

$$\left| \frac{\varphi_{2k-1}(x)}{\varphi_1(x)} \right| \leq c_5 (k - 3/4)^{11/6} \quad \text{and} \quad \left| \frac{\varphi_{2k}(x)}{\varphi_1(x)} \right| \leq c_5 (k - 1/2)^{11/6}.$$

for the ratio of the eigenfunctions, for every  $x \in \mathbb{R}$ . We have

$$|u(t, x, y) - e^{-\lambda_1 t} \varphi_1(x) \varphi_1(y)| \leq e^{-\lambda_1 t} \varphi_1(x) \varphi_1(y) e^{-(\lambda_2 - \lambda_1)t} \sum_{n=2}^{\infty} e^{-(\lambda_n - \lambda_2)t} \left| \frac{\varphi_n(x)}{\varphi_1(x)} \right| \left| \frac{\varphi_n(y)}{\varphi_1(y)} \right|.$$

For every  $t > 0$  the series on the right hand side is uniformly bounded by

$$\begin{aligned} \sum_{n=2}^{\infty} e^{-(\lambda_n - \lambda_2)t} \left| \frac{\varphi_n(x)}{\varphi_1(x)} \right| \left| \frac{\varphi_n(y)}{\varphi_1(y)} \right| &\leq c_5 \left( \sum_{k=2}^{\infty} \exp(-c_6(k - 3/4)^{2/3}) \left(k - \frac{3}{4}\right)^{\frac{11}{3}} \right. \\ &\quad \left. + \sum_{k=1}^{\infty} \exp(-c_7(k - 1/4)^{2/3}) \left(k - \frac{1}{4}\right)^{\frac{11}{3}} \right). \end{aligned}$$

Using the integral test for convergence it is easy to see that the above series converge. Thus the expression

$$e^{-(\lambda_2 - \lambda_1)t} \sum_{n=2}^{\infty} e^{-(\lambda_n - \lambda_2)t} \left| \frac{\varphi_n(x)}{\varphi_1(x)} \right| \left| \frac{\varphi_n(y)}{\varphi_1(y)} \right|$$

tends to zero uniformly in  $x, y \in \mathbb{R}$  as  $t \rightarrow \infty$ . This proves the estimates (3.13) for  $t > t_0$  and  $x \in \mathbb{R}$ ,  $y \in \mathbb{R}$  for some  $t_0 > 1$ .

Now let  $t \in [1, t_0]$ . Then we have

$$|u(t, x, y) - e^{-\lambda_1 t} \varphi_1(x) \varphi_1(y)| \leq c_8 e^{-\lambda_1 t} \varphi_1(x) \varphi_1(y)$$

with a constant  $c_8 > 0$ . This provides an upper bound for  $u(t, x, y)$ , for all  $x, y \in \mathbb{R}$ . Using the above estimates for the ratio  $\varphi_n(x)/\varphi_1(x)$ , dominated convergence and the asymptotic expressions for the eigenfunctions in Theorem 3.10 we obtain that for every  $t \in [1, t_0]$

$$\lim_{|x|, |y| \rightarrow \infty} \frac{u(t, x, y)}{\varphi_1(x) \varphi_1(y)} = -a'_1 \sum_{k=1}^{\infty} \frac{\exp(a'_k t)}{-a'_k} \geq -a'_1 \sum_{k=1}^{\infty} \frac{\exp(a'_k t_0)}{-a'_k} > 0.$$

The function  $e^{-\lambda_1 t}$  is comparable with a constant on  $[1, t_0]$ , whence

$$u(t, x, y) \geq c e^{-\lambda_1 t} \varphi_1(x) \varphi_1(y)$$

for  $t \in [1, t_0]$ ,  $|x| > x_0$  and  $|y| > y_0$  for suitable  $x_0, y_0 > 0$ .

Notice that for every  $a \in [-y_0, y_0]$  and  $s \in [1, t_0]$

$$\lim_{|x| \rightarrow \infty, y \rightarrow a, t \rightarrow s} \frac{u(t, x, y)}{\varphi_1(x) \varphi_1(y)} = \sqrt{-a'_1} \sum_{k=1}^{\infty} \frac{e^{a'_k s}}{\sqrt{-a'_k}} \frac{\varphi_{2k-1}(a)}{\varphi_1(a)} > 0.$$

Positivity of the limit is a consequence of the well-known general estimate  $u(t, x, y) \geq c_t \varphi_1(x) \varphi_1(y)$  derived from intrinsic ultracontractivity, with a constant  $c_t > 0$ . Thus there exist  $\varepsilon_s, \varepsilon_a > 0$ ,  $x_{s,a} > 0$  and a constant  $c_{s,a} > 1$  such that

$$\frac{1}{c_{s,a}} \leq \frac{u(t, x, y)}{\varphi_1(x) \varphi_1(y)} \leq c_{s,a}$$

for every  $(t, |x|, y) \in (s - \varepsilon_s, s + \varepsilon_s) \times (x_{s,a}, \infty) \times (a - \varepsilon_a, a + \varepsilon_a)$ . The family

$$\{(s - \varepsilon_s, s + \varepsilon_s) \times (a - \varepsilon_a, a + \varepsilon_a)\}_{(s,a) \in [1, t_0] \times [-y_0, y_0]}$$

is an open cover of the compact set  $[1, t_0] \times [-y_0, y_0]$ . Therefore there exists a finite subcover

$$\{(s_k - \varepsilon_{s_k}, s_k + \varepsilon_{s_k}) \times (a_k - \varepsilon_{a_k}, a_k + \varepsilon_{a_k})\}_{k=1,2,\dots,n}$$

of the set  $[1, t_0] \times [-y_0, y_0]$ . Putting  $c = \max\{c_{s_k, a_k} : k = 1, \dots, n\}$ ,  $x_1 = \max\{x_{s_k, a_k} : k = 1, \dots, n\}$  we get that

$$\frac{1}{c} \leq \frac{u(t, x, y)}{\varphi_1(x)\varphi_1(y)} \leq c$$

for every  $(t, |x|, y) \in [1, t_0] \times (x_1, \infty) \times [-y_0, y_0]$ . Due to the symmetry of  $u(t, x, y)$  we get the analogous result for  $(t, x, |y|) \in [1, t_0] \times [-x_0, x_0] \times [y_1, \infty)$ . Since  $u$  and  $\varphi_1$  are continuous and strictly positive we get

$$u(t, x, y) \geq ce^{-\lambda_1 t} \varphi_1(x)\varphi_1(y)$$

for  $(t, x, y) \in [1, t_0] \times [-\max\{x_0, x_1\}, \max\{x_0, x_1\}] \times [-\max\{y_0, y_1\}, \max\{y_0, y_1\}]$ . This completes the proof.  $\square$

## 4 Appendix: Airy functions

For the convenience of the reader we summarize some basic properties of Airy functions used in this paper.

The Airy functions  $\text{Ai}(x)$  and  $\text{Bi}(x)$  are defined as two independent solutions of the Airy equation

$$y'' - xy = 0, \quad x \in \mathbb{R}. \quad (4.1)$$

The equation can be easily reduced to the Bessel equation (for  $x \leq 0$ ) and to the modified Bessel equation ( $x > 0$ ). This allows to express the Airy functions in terms of Bessel functions  $J_\vartheta$  and modified Bessel functions  $K_\vartheta, I_\vartheta$  in the following way:

$$\text{Ai}(x) = \begin{cases} \frac{\sqrt{-x}}{3} \left[ J_{1/3} \left( \frac{2}{3}(-x)^{3/2} \right) + J_{-1/3} \left( \frac{2}{3}(-x)^{3/2} \right) \right], & x \leq 0, \\ \frac{1}{\pi} \sqrt{\frac{x}{3}} K_{1/3} \left( \frac{2}{3}x^{3/2} \right), & x > 0. \end{cases} \quad (4.2)$$

$$\text{Bi}(x) = \begin{cases} \sqrt{\frac{-x}{3}} \left[ J_{-1/3} \left( \frac{2}{3}(-x)^{3/2} \right) - J_{1/3} \left( \frac{2}{3}(-x)^{3/2} \right) \right], & x \leq 0, \\ \sqrt{\frac{x}{3}} \left[ I_{1/3} \left( \frac{2}{3}x^{3/2} \right) + I_{-1/3} \left( \frac{2}{3}x^{3/2} \right) \right], & x > 0. \end{cases} \quad (4.3)$$

Using the relation  $\text{Ai}''(x) = x\text{Ai}(x)$  we get that the  $n$ -th derivative of  $\text{Ai}$  is given by

$$\text{Ai}^{(n)}(x) = p_n(x)\text{Ai}(x) + q_n(x)\text{Ai}'(x), \quad (4.4)$$

where  $p_n$  and  $q_n$  are  $n$ th order polynomials defined by the recursive relations

$$\begin{aligned} p_{n+1}(x) &= p_n'(x) + xq_n(x), \\ q_{n+1}(x) &= p_n(x) + q_n'(x) \end{aligned}$$

and  $p_0(x) \equiv 1$ ,  $q_0(x) \equiv 0$ . Below we give formulae for  $p_n$  and  $q_n$  for  $n = 1, \dots, 10$ .

$$\begin{array}{ll}
p_1(x) = 0 & q_1(x) = 1 \\
p_2(x) = x & q_2(x) = 0 \\
p_3(x) = 1 & q_3(x) = x \\
p_4(x) = x^2 & q_4(x) = 2 \\
p_5(x) = 4x & q_5(x) = x^2 \\
p_6(x) = x^3 + 4 & q_6(x) = 6x \\
p_7(x) = 9x^2 & q_7(x) = x^3 + 10 \\
p_8(x) = x^4 + 28x & q_8(x) = 12x^2 \\
p_9(x) = 16x^3 + 28 & q_9(x) = x^4 + 52x \\
p_{10}(x) = x^5 + 100x^2 & q_{10}(x) = 20x^3 + 80
\end{array}$$

We recall some asymptotic results related to Airy functions. The asymptotic behaviour of Ai for large arguments is given by ([1], 10.4.59 and 10.4.60)

$$\text{Ai}(x) \cong \frac{1}{2\pi^{1/2}} x^{-1/4} e^{-\frac{2}{3}x^{3/2}}, \quad x \rightarrow \infty, \quad (4.5)$$

$$\text{Ai}(-x) = \frac{1}{\pi^{1/2}} \frac{\sin\left(\frac{2}{3}x^{3/2} + \pi/4\right)}{x^{1/4}} + O(x^{-7/4}), \quad x \rightarrow \infty. \quad (4.6)$$

The corresponding formula for Ai' is given by ([1], 10.4.61)

$$\text{Ai}'(x) \cong -\frac{1}{2\pi^{1/2}} x^{1/4} e^{-\frac{2}{3}x^{3/2}}, \quad x \rightarrow \infty. \quad (4.7)$$

Asymptotic formulae for Ai( $a'_k$ ) and Ai'( $a_k$ ) are ([1] 10.4.96 and 10.4.97)

$$\text{Ai}(a'_k) \sim (-1)^{k-1} \pi^{-1/2} \left(\frac{3\pi}{2}\right)^{-1/6} (k - 3/4)^{-1/6}, \quad k \rightarrow \infty, \quad (4.8)$$

$$\text{Ai}'(a_k) \sim (-1)^{k-1} \pi^{-1/2} \left(\frac{3\pi}{2}\right)^{1/6} (k - 1/2)^{1/6}, \quad k \rightarrow \infty. \quad (4.9)$$

Finally we prove a result we have used in the previous section.

**Lemma 4.1.** *For any open set  $D \subset \mathbb{R}$  and  $a > 0$  let  $u \mapsto f(x, u)$  be a continuous function for every fixed  $x \in D$  such that  $\int_a^y f(x, u) du$  is uniformly bounded for  $(x, y) \in D \times [a, \infty)$ . Assume that  $(x, u) \mapsto g(x, u)$  is a uniformly bounded function on  $D \times [a, \infty)$  such that for every  $x \in D$  and  $y > a$  the function  $u \rightarrow g(x, u)$  has continuous derivative on  $[a, y)$ , and its derivative has at most one zero in  $[a, y)$ . Then*

$$F(x, y) = \int_a^y f(x, u) g(x, u) du$$

is a uniformly bounded function in  $D \times [a, \infty)$ .

*Proof.* Denote by  $b$  the zero of the function  $g(x, \cdot)$  in the interval  $[a, y)$ . If there are no zeroes of  $g(x, \cdot)$  in this interval  $b$  can be arbitrarily chosen in the interval. Fix  $x \in D$ . Applying the second mean value theorem for integration to the intervals  $[a, b]$  and  $[b, y]$  apart it follows that there exist constants  $\lambda_1 \in [a, b]$  and  $\lambda_2 \in [b, y]$  such that  $F(x, y)$  is equal to

$$g(x, a) \int_a^{\lambda_1} f(x, u) du + g(x, b) \int_{\lambda_1}^b f(x, u) du + g(x, b) \int_b^{\lambda_2} f(x, u) du + g(x, y) \int_{\lambda_2}^y f(x, u) du.$$

Using the assumption that the functions  $g(x, u)$  and  $\int_a^y f(x, u) du$  are uniformly bounded, the result follows.  $\square$

**Acknowledgments:** It is a pleasure to thank T. Byczkowski, T. Kulczycki and A. Strohmaier for discussions.

## References

- [1] M. Abramowitz and I.A. Stegun: *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, 1972
- [2] K. Bogdan et al: *Potential Theory of Stable Processes and its Extensions*, LNM **1980**, Springer, 2009
- [3] N.H. Bingham: Maxima of sums of random variables and suprema of stable processes, *Z. Wahrscheinlichkeitstheorie verw. Geb.* **26**, 273-296 (1973)
- [4] D.A. Darling: The maximum of sums of stable random variables, *Trans. AMS* **83** (1956), 164-169
- [5] J. Dong and M. Xu: Some solutions to the space fractional Schrödinger equation using momentum representation method, *J. Math. Phys.* **48**, 072105 (2007)
- [6] H. W. Heathcote: Bounds for zeros of some special functions, *Proc. AMS* **25** (1970), 72-74
- [7] F. Hiroshima, T. Ichinose and J. Lörinczi: Path integral representation for Schrödinger operators with Bernstein functions of the Laplacian, *arXiv 0906.0103*, 2009
- [8] N. Jacob: *Pseudo-Differential Operators and Markov Processes: Markov Processes and Applications*, vols. 1-3, Imperial College Press, 2003-2005
- [9] K. Kaleta and T. Kulczycki: Intrinsic ultracontractivity for Schrödinger operators based on fractional Laplacian, preprint, 2009
- [10] K. Kaleta and J. Lörinczi: Analytic properties of fractional Schrödinger semigroups and Gibbs measures for symmetric stable processes, preprint, 2010
- [11] M. Kwasnicki, T. Kulczycki, J. Małecki and A. Stos: Spectral properties of the Cauchy process on half-line and interval, preprint 2009
- [12] N. Laskin, Fractional quantum mechanics, *Phys. Rev.* **E62** (2000), 3135
- [13] J. Lörinczi, F. Hiroshima and V. Betz: *Feynman-Kac-Type Theorems and Gibbs Measures on Path Space. With Applications to Rigorous Quantum Field Theory*, de Gruyter Studies in Mathematics **34**, Walter de Gruyter, Berlin-New York, to appear, 2010
- [14] G. Pittaluga and L. Sacripante: Inequalities for the zeros of the Airy functions, *SIAM J. Math. Anal.* **22** (1991), 260-267
- [15] D. Revuz and M. Yor: *Continuous Martingales and Brownian Motion*, 3rd ed., Springer (New York), 1999
- [16] B. Simon: *Functional Integration and Quantum Physics*, 2nd ed., AMS Chelsea Publishing, 2004
- [17] O. Vallée and M. Soares, *Airy Functions and Applications to Physics*, Imperial College Press, 2004