Local Time Rough Path for Lévy Processes

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Summary. In this paper, we will prove that the local time of a Lévy process is a rough path of roughness p a.s. for any $2 < p < 3$ under some condition for the Lévy measure. This is a new class of rough path processes. Then for any function g of finite q-variation ($1 \leq q < 3$), we establish the integral $\int_{-\infty}^{\infty} g(x) dL_t^x$ as a Young integral when $1 \leq q < 2$ and a Lyons' rough path integral when $2 \leq q < 3$. We therefore apply these path integrals to extend the Tanaka-Meyer formula for a continuous function f if f'_- exists and is of finite q-variation when $1 \leq q < 3$, for both continuous semi-martingales and a class of Lévy processes.

Keywords: semimartingale local time; geometric rough path; Young integral; rough path integral; Lévy processes.

1 Introduction

Integration of a stochastic process is a fundamentally important problem in probability theory. Different integration theory may result in completely different approach to a problem. Local time is an important and useful stochastic process. The investigation of its variation and integration has attracted attentions of many mathematicians. Similar to the case of the Brownian motion, the variation of the local time of a semimartingale in the spatial variable is also fundamental in the construction of an integral with respect to the local time. There have been many works on the quadratic or p-variations (in the case of stable processes) of local times in the sense of probability. Bouleau and Yor ([3]), Perkins ([28]), first proved that, for the Brownian local time, and a sequence of partitions $\{D_n\}$ of an interval $[a, b]$, with the mesh $|D_n| \to 0$ when $n \to \infty$, $\lim_{n\to\infty} \sum_{D_n} (L_t^{x_{i+1}} - L_t^{x_i})^2 = 4 \int_a^b L_t^x dx$ in probability. Subsequently, the process $x \to L_t^x$ can be regarded as a semimartingale (with appropriate filtration). This result allowed one to construct various stochastic integrals of the Brownian local time in the spatial variable. See also Rogers and Walsh [31]. Numerous important extensions on the variations, stochastic integrations of local times and Itô's formula have been done, e.g. Marcus and Rosen [24], [25], Eisenbaum [5], [6], Eisenbaum and Kyprianou [7], Flandoli, Russo and Wolf [11], Föllmer, Protter and Shiryayev [12], Föllmer and Protter [13], Moret and Nualart [27]. In their extensions of Itô's formula, the integrals of the local time are given as stochastic integrals in nature, for example as forward and backward stochastic integrals. In this paper, we study path integration of the local time and prove that the local time is of classical p-variation and also can be considered as a rough path (its meaning will be made precise later). We consider the local time of the Lévy process which is represented by the following Lévy-Itô decomposition

$$X_t = X_0 + \sigma B_t + bt + \int_0^{t+} \int_{R\setminus\{0\}} y 1_{\{|y|\geq 1\}} N_p(dsdy) + \int_0^{t+} \int_{R\setminus\{0\}} y 1_{\{|y|<1\}} \tilde{N}_p(dsdy). \quad (1.1)$$

Recall that for a general semimartingale X_t, $L=\{L_t^x; x\in R\}$ is defined from the following formula (Meyer [26]):

$$\int_0^t g(X_s)d[X,X]_s^c=\int_{-\infty}^{\infty} g(x)L_t^x dx, \tag{1.2}$$

where $[X,X]_\cdot^c$ is the continuous part of the quadratic variation process $[X,X]_\cdot$. There is a different notion of local time defined as the Radon-Nikodym derivative of the occupation measure of X with respect to the Lebesgue measure on R i.e.

$$\int_0^t g(X_s)ds=\int_{-\infty}^{\infty} g(x)\gamma_t^x dx, \tag{1.3}$$

for every Borel function $g: R\to R^+$. For the Lévy process (1.1), if $\sigma\neq 0$, L_t^x and γ_t^x are the same (up to a multiple of a constant). In case $\sigma=0$ e.g. for a stable process, there is no diffusion part so these two definitions are different. In fact, in this case $L_t^x=0$. The increment of $\gamma_t^\cdot$ for stable processes was considered by Boylan [4], Getoor and Kesten [14] and Barlow [2], using potential theory approach, in order to establish the continuity of the local time in the space variable. Using Theorem 0.2 in [2], it's easy to prove that when $\sigma\neq 0$, and (2.2) is satisfied, for any $p>2$ and $t\geq 0$, the process $x\mapsto L_t^x$, is of finite p-variation in the classical sense almost surely. As a direct application, one can define the path integral $\int_{-\infty}^{+\infty} g(x)dL_t^x$ as a Young integral for any g being of bounded q-variation for a $q\in[1,2)$. But when $q\geq 2$, Young's condition $\frac{1}{p}+\frac{1}{q}>1$ is broken.

The main task of this paper is to construct a geometric rough path over the processes $Z(x)=(L_t^x, g(x))$, for a deterministic function g being of finite q-variation when $q\in[2,3)$. This implies establishing the path integrals $\int_{-\infty}^{\infty} L_t^x d_x L_t^x$ and $\int_{-\infty}^{\infty} g(x)d_x L_t^x$. For these two integrals, all classical integration theories such as Riemann, Lebesgue and Young fail to work. To overcome the difficulty, we use the rough path theory pioneered by Lyons, see [21], [22], [23], also [19]. However, our p-variation result of the local time does not automatically make the desired rough path exist or the integral well defined, though it is a crucial step to study first. Actually further hard analysis is needed to establish an iterated path integration theory for $Z_\cdot$. First we introduce a piecewise curve of bounded variation as a generalized Wong-Zakai approximation to the stochastic process $Z_\cdot$. Then we define a smooth rough path by defining the iterated integrals of the piecewise bounded variation process. We need to prove the smooth rough path converges to a geometric rough path $\mathbf{Z}=(1,\mathbf{Z}^1,\mathbf{Z}^2)$ when $2\leq q<3$. For this, an important step is to compute $E(L_t^{x_{i+1}}-L_t^{x_i})(L_t^{x_{j+1}}-L_t^{x_j})$, and obtain the correct order in terms of the increments $x_{i+1}-x_i$ and $x_{j+1}-x_j$, especially in disjoint intervals $[x_i,x_{i+1}]$ and $[x_j,x_{j+1}]$ when $i\neq j$. One can see the global aspects of Lévy processes are captured in this estimate. Actually, this is a very challenging task. In this analysis, the main difficulty is from dealing with jumps, especially the small jumps of the process. One can also see that to construct the geometric rough path, a slightly stronger condition (3.24) is needed here. Using this key estimate, we can establish the geometric rough path $\mathbf{Z}=(1,\mathbf{Z}^1,\mathbf{Z}^2)$. Then from Chen's identity, we define the following two integrals

$$\int_a^b L_t^x dL_t^x=\lim_{m(D_{[a,b]})\to 0}\sum_{i=0}^{r-1}((\mathbf{Z}^2_{x_i,x_{i+1}})_{1,1}+L_t(x_i)(L_t^{x_{i+1}}-L_t^{x_i})) \tag{1.4}$$

and

$$\int_a^b g(x)dL_t^x=\lim_{m(D_{[a,b]})\to 0}\sum_{i=0}^{r-1}((\mathbf{Z}^2_{x_i,x_{i+1}})_{2,1}+g(x_i)(L_t^{x_{i+1}}-L_t^{x_i})). \tag{1.5}$$

Note that the Riemann sum $\sum_{i=0}^{r-1} L_t^{x_i}(L_t^{x_{i+1}} - L_t^{x_i})$ and $\sum_{i=0}^{r-1} g(x_i)(L_t^{x_{i+1}} - L_t^{x_i})$ themselves may not have limits as the mesh $m(D_{[a,b]}) \to 0$. At least there are no integration theories, rather than Lyons' rough path theory, to guarantee the convergence of the Riemann sums for almost all ω. Here it is essential to add Lévy areas to the Riemann sum. Furthermore, we can prove if a sequence of smooth functions $g_j \to g$ as $j \to \infty$, then the Riemann integral $\int_a^b g_j(x) dL_t^x$ converges to the rough path integral $\int_a^b g(x) dL_t^x$ defined in (1.5). It is also noted that to establish (1.4), one only needs (3.25). This is true as long as the power of $|y|$ in the condition of Lévy measure is anything less (better) than $\frac{3}{2}$. The main technique here is the Tanaka formula. If we assume the processes is symmetric, one can use estimates of Gaussian processes and Dynkin's isomorphism theorem as our tool. This idea is being developed by Wang ([33]) for symmetric stable processes.

Having established the path integration of local time and the corresponding convergence results, as a simple application, we can easily prove a useful extension of Itô's formula for the Lévy process when the function is less smooth: if $f : R \to R$ is an absolutely continuous function and has left derivative $f'_-(x)$ being left continuous and of bounded q-variation, where $1 \le q < 3$, then P-a.s.

$$
\begin{aligned}
f(X_t) = f(X_0) + \int_0^t f'_-(X_s) dX_s - \int_{-\infty}^{\infty} f'_-(x) d_x L_t^x \\
+ \sum_{0 \le s \le t} [f(X_s) - f(X_{s-}) - \Delta X_s f'_-(X_{s-})], \quad 0 \le t < \infty. \qquad (1.6)
\end{aligned}
$$

Here the path integral $\int_{-\infty}^{\infty} f'_-(x) d_x L_t^x$ is a Lebesgue-Stieltjes integral when $q = 1$, a Young integral when $1 < q < 2$ and a Lyons' rough path integral when $2 \le q < 3$ respectively. Needless to say that Tanaka's formula ([32]) and Meyer's formula ([26], [34]) are special cases of our formula when $q = 1$. The investigation of Itô's formula to less smooth functions is crucial and useful in many problems e.g. studying partial differential equations with singularities, the free boundary problem in American options, and certain stochastic differential equations. Time dependent cases for a continuous semimartingale X_t were investigated recently by Elworthy, Truman and Zhao ([8]), Feng and Zhao ([9]), where two-parameter Lebesgue-Stieltjes integrals and two-parameter Young integral were used respectively. We would like to point out that a two-parameter rough path integration theory, which is important to the study of local times, and some other problems such as SPDEs, still remains open.

A part of the results about the rough path integral of local time for a continuous semimartingale was announced without including proofs in Feng and Zhao [10]. In this paper, we will give a full construction of the local time rough path, and obtain complete results for any continuous semi-martingales and a class of Lévy processes satisfying (3.24) and $\sigma \neq 0$. Our proofs are given in the context of Lévy processes. For continuous semimartingales, we believe the reader can see easily that the proof is essentially included in this paper, noticing the idea of decomposing the local time to continuous and discontinuous parts in [9] and the key estimate (8) in [10].

2 The variation of local time

We see soon that the variation of local time follows immediately from Barlow's celebrated result of modulus of continuity of local time (Theorem 0.2, [2]). Let X_t be a one dimensional time homogeneous Lévy process, and $(\mathcal{F}_t)_{t \ge 0}$ be generated by the sample paths X_t, $p(\cdot)$ be a stationary $(\mathcal{F}_t)$-Poisson process on $R \setminus \{0\}$. From the well-known Lévy-Itô decomposition theorem, we can write X_t as follows:

$$X_t := X_0 + \sigma B_t + V_t + \tilde{M}_t, \tag{2.1}$$

where

$$V_t := bt + \int_0^{t+} \int_{R\setminus\{0\}} y1_{\{|y|\geq 1\}} N_p(dsdy),$$
$$\tilde{M}_t := \int_0^{t+} \int_{R\setminus\{0\}} y1_{\{|y|<1\}} \tilde{N}_p(dsdy).$$

Here, N_p is the Poisson random measure of p, the compensator of p is of the form $\hat{N}(dsdy) = dsn(dy)$, where $n(dy)$ is the Lévy measure of process X. The compensated random measure $\tilde{N}_p(t,U) = N_p(t,U) - \hat{N}_p(t,U)$ is an $(\mathcal{F}_t)$-martingale.

Proposition 2.1 *If $\sigma \neq 0$, the Lévy measure n satisfies*

$$\int_{R\setminus\{0\}} (|y|^2 \wedge 1) n(dy) < \infty, \tag{2.2}$$

then the local time L_t^a of time homogeneous Lévy process X_t given by (1.1) is of bounded p-variation in a for any $t \geq 0$, for any $p > 2$, almost surely, i.e.

$$\sup_{D(-\infty,\infty)} \sum_i |L_t^{a_{i+1}} - L_t^{a_i}|^p < \infty \quad a.s.,$$

where the supremum is taken over all finite partition on R, $D(-\infty,\infty) := \{-\infty < a_0 < a_1 < \cdots < a_n < \infty\}$.

Proof: Let $\chi(\theta)$ be the exponent of Lévy process X, i.e.

$$Ee^{i\theta X_t} = e^{-t\chi(\theta)},$$

where

$$\chi(\theta) = -ib\theta + \sigma^2\theta^2 - \int [e^{i\theta y} - 1 - i\theta y 1_{\{|y|<1\}}] n(dy). \tag{2.3}$$

Recall from [2],

$$\phi(x)^2 = \frac{1}{\pi} \int (1 - \cos\theta x) Re(\frac{1}{1+\chi(\theta)}) d\theta$$

If $\sigma^2 > 0$, it is easy to check that

$$Re(\frac{1}{1+\chi(\theta)}) \leq c(\sigma,\theta)(1+\theta^2)^{-1},$$

and $\phi(x) \leq c|x|^{\frac{1}{2}}$. So using Theorem 0.2 in [2], one obtains that if $\rho(x) = (xlog(\frac{1}{x}))^{\frac{1}{2}}$

$$\sup_{|b-a|\leq\frac{1}{2}} |L_t^a - L_t^b| \leq c\rho(|b-a|)(\sup_x L_t^x)^{\frac{1}{2}}. \tag{2.4}$$

Now we use Proposition 4.1.1 in [23] ($i = 1, \gamma > p-1$), for any finite partition $\{a_l\}$ of $[a,b]$

$$\sup_D \sum_l |L_t^{a_{l+1}} - L_t^{a_l}|^p \leq c(p,\gamma) \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} |L_t^{a_k^n} - L_t^{a_{k-1}^n}|^p,$$

where

$$a_k^n = a + \frac{k}{2^n}(b-a),\ \ k = 0, 1, \cdots, 2^n.$$

The key point here is that the right hand side does not depend on the partition D. Using (2.4), we know that

$$\sum_{n=1}^{\infty} n^{\gamma} \sum_{k=1}^{2^n} |L_t^{a_k^n} - L_t^{a_{k-1}^n}|^p \le c \sum_{n=1}^{\infty} n^{\gamma} (\rho(\frac{b-a}{2^n}))^{\frac{p}{2}}$$
$$\le \tilde{c} \sum_{n=1}^{\infty} n^{\gamma} (\frac{b-a}{2^n})^{\frac{p}{2}-1-\varepsilon} < \infty,$$

as $p > 2$, where $\varepsilon < \frac{p}{2} - 1$. Therefore for any interval $[a, b] \subset R$

$$\sup_D \sum_l |L_t^{a_{l+1}} - L_t^{a_l}|^p < \infty \ a.s. \tag{2.5}$$

But we know L_t^a has a compact support $[-K, K]$ in a. So for the partition $D := D_{-K,K} = \{-K = a_0 < a_1 < \cdots < a_r = K\}$, we obtain

$$\sup_D \sum_l |L_t^{a_{i+1}} - L_t^{a_i}|^p < \infty \ a.s. \qquad \diamond$$

The p-variation ($p > 2$) result of the local time enables one to use Young's integration theory to define $\int_{-\infty}^{\infty} g(x) d_x L_t^x$ for g being of bounded q-variation when $1 \le q < 2$. This is because in this case, for any $q \in [1, 2)$, one can always find a constant $p > 2$ such that the condition $\frac{1}{p} + \frac{1}{q} > 1$ for the existence of the Young integral is satisfied. However, when $q \ge 2$, Young integral is no longer well defined. We have to use a new integration theory. In the following, we only consider the case that $2 \le q < 3$. We obtain the existence of the geometric rough path $\mathbf{Z} = (1, \mathbf{Z}^1, \mathbf{Z}^2)$ associated to $Z_{\cdot}$. Lyons' integration of rough path provides a way to push the result further. This will be studied in the rest of this paper.

3 The local time rough path

To establish the rough path integral of local time, we need to estimate the p-th moment of the increment of local time over a space interval and the covariance of the increments over two nonoverlapping intervals. First, we give a general p-moment estimate formula. This will be used in later proofs.

Lemma 3.1 *We have the p-moment estimate formula: for any $p \ge 1$,*

$$E\Big(\sum_{0 \le s \le t} |f(s, p_s(\omega), \omega)|\Big)^p$$
$$\le c_p \sum_{k=0}^{m} E\Big(\int_0^t \int_R |f(s, y, \omega)|^{2^k} n(dy) ds\Big)^{\frac{p}{2^k}} + c_p \Big(E \int_0^t \int_R |f(s, y, \omega)|^{2^{m+1}} n(dy) ds\Big)^{\frac{p}{2^{m+1}}}, \tag{3.1}$$

for a constant $c_p > 0$. Here m is the smallest integer such that $2^{m+1} \ge p$.

Proof: From the definition of N_p, $\tilde{N}_p$, the Burkholder-Davis-Gundy inequality and Jensen's inequality, we can have the p-moment estimation:

$$
\begin{aligned}
& E\Big(\sum_{0\leq s\leq t}|f(s,p_s(\omega),\omega)|\Big)^p \\
&= E\Big(\int_0^{t+}\int_{R\setminus\{0\}}|f(s,y,\omega)|N_p(dsdy)\Big)^p \\
&= E\Big(\int_0^t\int_R|f(s,y,\omega)|n(dy)ds+\int_0^{t+}\int_{R\setminus\{0\}}|f(s,y,\omega)|\big(N_p(dsdy)-n(dy)ds\big)\Big)^p \\
&\leq pE\Big(\int_0^t\int_R|f(s,y,\omega)|n(dy)ds\Big)^p+pE\Big(\int_0^{t+}\int_{R\setminus\{0\}}|f(s,y,\omega)|\tilde{N}_p(dsdy)\Big)^p \\
&\leq pE\Big(\int_0^t\int_R|f(s,y,\omega)|n(dy)ds\Big)^p+c_pE\Big(\int_0^t\int_R|f(s,y,\omega)|^2n(dy)ds\Big)^{\frac{p}{2}} \\
&\quad +c_pE\Big(\int_0^t\int_{R\setminus\{0\}}|f(s,y,\omega)|^2\tilde{N}_p(dsdy)\Big)^{\frac{p}{2}} \\
&\leq pE\Big(\int_0^t\int_R|f(s,y,\omega)|n(dy)ds\Big)^p+c_pE\Big(\int_0^t\int_R|f(s,y,\omega)|^2n(dy)ds\Big)^{\frac{p}{2}} \\
&\quad +\cdots+c_pE\Big(\int_0^t\int_R|f(s,y,\omega)|^{2^m}n(dy)ds\Big)^{\frac{p}{2^m}}+c_p\left(E\Big(\int_0^t\int_{R\setminus\{0\}}|f(s,y,\omega)|^{2^m}\tilde{N}_p(dsdy)\Big)^2\right)^{\frac{p}{2^{m+1}}} \\
&= pE\Big(\int_0^t\int_R|f(s,y,\omega)|n(dy)ds\Big)^p+c_pE\Big(\int_0^t\int_R|f(s,y,\omega)|^2n(dy)ds\Big)^{\frac{p}{2}} \\
&\quad +\cdots+c_pE\Big(\int_0^t\int_R|f(s,y,\omega)|^{2^m}n(dy)ds\Big)^{\frac{p}{2^m}}+c_p\Big(E\int_0^t\int_R|f(s,y,\omega)|^{2^{m+1}}n(dy)ds\Big)^{\frac{p}{2^{m+1}}},
\end{aligned}
$$

where m is the smallest integer such that $2^{m+1}\geq p$. ◇

Recall the Tanaka formula for the Lévy process X_t (c.f. [1]), we have

$$
\begin{aligned}
L_t^a &= (X_t-a)^+-(X_0-a)^+-\int_0^t 1_{\{X_{s-}>a\}}dX_s \\
&\quad +\sum_{0\leq s\leq t}[(X_{s-}-a)^+-(X_s-a)^++1_{\{X_{s-}>a\}}\Delta X_s].
\end{aligned} \tag{3.2}
$$

Since

$$
\sum_{0\leq s\leq t}1_{\{X_{s-}>a\}}\Delta X_s 1_{\{|\Delta X_s|\geq 1\}}=\int_0^{t+}\int_{R\setminus\{0\}}1_{\{X_{s-}>a\}}y1_{\{|y|\geq 1\}}N_p(dsdy),
$$

the following alternative form is often convenient

$$
L_t^a=\varphi_t(a)-bI_t(a)-\sigma\hat{B}_t^a+K_1(t,a)+K_2(t,a)+K_3(t,a), \tag{3.3}
$$

where

$$
\varphi_t(a):=(X_t-a)^+-(X_0-a)^+,\quad I_t(a):=\int_0^t 1_{\{X_{s-}>a\}}ds,\quad \hat{B}_t^a:=\int_0^t 1_{\{X_{s-}>a\}}dB_s,
$$

$$K_1(t,a) := \sum_{0\le s\le t}[(X_{s-}-a)^+ - (X_s-a)^+]1_{\{|\Delta X_s|\ge 1\}},$$
$$K_2(t,a) := \int_0^t\int_{R\setminus\{0\}}[(X_s-y-a)^+ - (X_s-a)^+]1_{\{|y|<1\}}\tilde{N}_p(dyds),$$
$$K_3(t,a) := \int_0^t\int_R[(X_s-y-a)^+ - (X_s-a)^+ + 1_{\{X_s-y>a\}}y]1_{\{|y|<1\}}n(dy)ds.$$

For the convenience in what follows in later part, we denote

$$J_1(s,a) := (X_{s-}-a)^+ - (X_s-a)^+, \quad J_2(s,a) := 1_{\{X_{s-}>a\}}\Delta X_s.$$

Note we have the following important decompositions that we will use often: for any $a_i < a_{i+1}$,

$$\begin{aligned}
& J_*(X_s, X_{s-}, a_i, a_{i+1}) \\
:= & J_1(s,a_{i+1}) - J_1(s,a_i) \\
= & -(X_{s-}-a_i)1_{\{X_s\le a_i\}}1_{\{a_i<X_{s-}\le a_{i+1}\}} - (a_{i+1}-a_i)1_{\{X_s\le a_i\}}1_{\{X_{s-}>a_{i+1}\}} \\
& +(X_s-a_i)1_{\{a_i<X_s\le a_{i+1}\}}1_{\{X_{s-}\le a_i\}} - (a_{i+1}-X_s)1_{\{a_i<X_s\le a_{i+1}\}}1_{\{X_{s-}>a_{i+1}\}} \\
& +(a_{i+1}-a_i)1_{\{X_s>a_{i+1}\}}1_{\{X_{s-}\le a_i\}} + (a_{i+1}-X_{s-})1_{\{X_s>a_{i+1}\}}1_{\{a_i<X_{s-}\le a_{i+1}\}} \\
& +(X_s-X_{s-})1_{\{a_i<X_s\le a_{i+1}\}}1_{\{a_i<X_{s-}\le a_{i+1}\}},
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
& J^*(X_s, X_{s-}, a_i, a_{i+1}) \\
:= & [J_1(s,a_{i+1}) - J_1(s,a_i)] + [J_2(s,a_{i+1}) - J_2(s,a_i)] \\
= & -(X_s-a_i)1_{\{X_s\le a_i\}}1_{\{a_i<X_{s-}\le a_{i+1}\}} - (a_{i+1}-a_i)1_{\{X_s\le a_i\}}1_{\{X_{s-}>a_{i+1}\}} \\
& +(X_s-a_i)1_{\{a_i<X_s\le a_{i+1}\}}1_{\{X_{s-}\le a_i\}} - (a_{i+1}-X_s)1_{\{a_i<X_s\le a_{i+1}\}}1_{\{X_{s-}>a_{i+1}\}} \\
& +(a_{i+1}-a_i)1_{\{X_s>a_{i+1}\}}1_{\{X_{s-}\le a_i\}} + (a_{i+1}-X_s)1_{\{X_s>a_{i+1}\}}1_{\{a_i<X_{s-}\le a_{i+1}\}}.
\end{aligned} \tag{3.5}$$

We will use the above decomposition and probabilistic tools to prove the following lemma. The proof includes many technically rather hard calculations. However, they are key to get the desired estimates and crucial in our analysis.

Lemma 3.2 *Assume the Lévy measure $n(dy)$ satisfies*

$$\int_{R\setminus\{0\}}(|y|^{\frac{3}{2}}\wedge 1)n(dy) < \infty, \tag{3.6}$$

then for any $p\ge 2$, there exists a constant $c>0$ such that

$$E|L_t^{a_{i+1}} - L_t^{a_i}|^p \le c|a_{i+1}-a_i|^{\frac{p}{2}}. \tag{3.7}$$

Proof: We will estimate every term in (3.3). First note that the function $\varphi_t(a) := (X_t-a)^+ - (X_0-a)^+$ is Lipschitz continuous in a with Lipschitz constant 2. This implies that for any $p>2$ and $a_i<a_{i+1}$,

$$|\varphi_t(a_{i+1}) - \varphi_t(a_i)|^p \le 2^p(a_{i+1}-a_i)^p. \tag{3.8}$$

Secondly, for the second term, by the occupation times formula, Jensen's inequality and Fubini theorem,

$$
\begin{aligned}
E|I_t(a_{i+1})-I_t(a_i)|^p &= \frac{1}{\sigma^{2p}}(a_{i+1}-a_i)^p E\Big(\frac{1}{a_{i+1}-a_i}\int_{a_i}^{a_{i+1}} L_t^x dx\Big)^p \\
&\le \frac{1}{\sigma^{2p}}(a_{i+1}-a_i)^p \sup_x E(L_t^x)^p. \qquad (3.9)
\end{aligned}
$$

We now estimate $\sup_x E(L_t^x)^p$. By (3.2) and noting that $\sum_{0\le s\le t}[(X_{s-}-a)^+-(X_s-a)^+ + 1_{\{X_{s-}>a\}}\Delta X_s]$ is a decreasing process in t, we have

$$
L_t^a \le (X_t-a)^+ - (X_0-a)^+ - \int_0^t 1_{\{X_{s-}>a\}} dX_s.
$$

Now using the Burkholder-Davis-Gundy inequality, we have

$$
\begin{aligned}
E(L_t^a)^p &\le p\Big[E|X_t-X_0|^p + E\Big|\int_0^t \sigma 1_{\{X_{s-}>a\}} dB_s\Big|^p + E\Big|\int_0^t 1_{\{X_{s-}>a\}} dV_s\Big|^p + E\Big|\int_0^t 1_{\{X_{s-}>a\}} d\tilde{M}_s\Big|^p\Big] \\
&\le c\sigma^p t^{\frac{p}{2}} + cE\Big(\int_0^t |dV_s|\Big)^p + c_p \sum_{k=1}^{m+1}\Big[\int_0^t\int_R |y|^{2^k} 1_{\{|y|<1\}} n(dy)ds\Big]^{\frac{p}{2^k}} \\
&\le c(p,b,\sigma,t),
\end{aligned}
$$

where $m>0$ is the smallest integer such that $2^{m+1}\ge p$, $c(p,b,\sigma,t)$ is a universal constant depending on p,b,σ, and t. By Jensen's inequality, we also have

$$
E(L_t^a)^{\frac{p}{2}} \le c(p,b,\sigma,t). \qquad (3.10)
$$

This inequality will be used later. So

$$
E|I_t(a_{i+1})-I_t(a_i)|^p \le c(p,b,\sigma,t)(a_{i+1}-a_i)^p. \qquad (3.11)
$$

Thirdly, for the term $\hat{B}_t^a$, by the Burkholder-Davis-Gundy inequality and a similar argument in deriving (3.11), we have

$$
\begin{aligned}
E|\hat{B}_t^{a_{i+1}}-\hat{B}_t^{a_i}|^p &\le c_p E\left(\int_0^t 1_{\{a_i<X_s\le a_{i+1}\}} ds\right)^{\frac{p}{2}} \\
&\le c(t,p,\sigma)(a_{i+1}-a_i)^{\frac{p}{2}}. \qquad (3.12)
\end{aligned}
$$

About $K_1(t,a)$, it is easy to see that

$$
|K_1(t,a_{i+1})-K_1(t,a_i)| \le 2(a_{i+1}-a_i)\sum_{0\le s\le t} 1_{\{|\Delta X_s|\ge 1\}}.
$$

So

$$
E|K_1(t,a_{i+1})-K_1(t,a_i)|^p \le C(a_{i+1}-a_i)^p. \qquad (3.13)
$$

About $K_2(t,a)$, with the decomposition (3.4), we will estimate the sum of each term for jumps $|\Delta X_s|<1$. There are seven such terms.

For the first term in (3.4), by the p-moment estimate formula and occupation times formula, we have

$$
\begin{aligned}
&E\left(\int_0^t\int_{R\setminus\{0\}}|X_{s-}-a_i|1_{\{X_s\le a_i\}}1_{\{a_i<X_{s-}\le a_{i+1}\}}1_{\{|\Delta X_s|<1\}}\tilde N_p(dyds)\right)^p\\
&\le c_p\sum_{k=1}^m E\left(\int_0^t\int_{X_s-a_{i+1}}^{X_s-a_i}|X_s-y-a_i|^{2^k}1_{\{X_s\le a_i\}}1_{\{|y|<1\}}n(dy)ds\right)^{\frac{p}{2^k}}\\
&\quad +c_p\left(E\int_0^t\int_{X_s-a_{i+1}}^{X_s-a_i}|X_s-y-a_i|^{2^{m+1}}1_{\{X_s\le a_i\}}1_{\{|y|<1\}}n(dy)ds\right)^{\frac{p}{2^{m+1}}}\\
&= c(p,\sigma)\sum_{k=1}^m E\left(\int_{-\infty}^{a_i}L_t^x\int_{x-a_{i+1}}^{x-a_i}|x-y-a_i|^{2^k}1_{\{|y|<1\}}n(dy)dx\right)^{\frac{p}{2^k}}\\
&\quad +c(p,\sigma)\left(E\int_{-\infty}^{a_i}L_t^x\int_{x-a_{i+1}}^{x-a_i}|x-y-a_i|^{2^{m+1}}1_{\{|y|<1\}}n(dy)dx\right)^{\frac{p}{2^{m+1}}}, \qquad (3.14)
\end{aligned}
$$

where m is the smallest integer such that $2^{m+1}\ge p$. In the following we will often use the following type of method to estimate integrals with respect to the Lévy measure: let

$$
\begin{aligned}
Q &:= \int_{a_i-a_{i+1}}^0\int_{y+a_i}^{a_i}(x-y-a_i)dx1_{\{|y|<1\}}n(dy)\\
&= \frac12\int_{a_i-a_{i+1}}^0|y|^2 1_{\{|y|<1\}}n(dy).
\end{aligned}
$$

Then, $1_{\{|y|<1\}}\frac{1}{Q}(x-y-a_i)dxn(dy)$ is a probability measure on $\{(x,y): y+a_i\le x\le a_i, (-1)\wedge(a_i-a_{i+1})\le y\le 0\}$. So by Jensen's inequality, we have that

$$
\begin{aligned}
&E\left(\int_{a_i-a_{i+1}}^0\int_{y+a_i}^{a_i}L_t^x(a_{i+1}-a_i)(x-y-a_i)dx1_{\{|y|<1\}}n(dy)\right)^{\frac p2}\\
&\le Q^{\frac p2}E\left(\frac1Q\int_{a_i-a_{i+1}}^0\int_{y+a_i}^{a_i}(L_t^x)^{\frac p2}(a_{i+1}-a_i)^{\frac p2}(x-y-a_i)dx1_{\{|y|<1\}}n(dy)\right)\\
&\le c(\sigma,t,p)|a_{i+1}-a_i|^{\frac p2}\sup_x E(L_t^x)^{\frac p2}\left(\int_{a_i-a_{i+1}}^0|y|^2 1_{\{|y|<1\}}n(dy)\right)^{\frac p2}. \qquad (3.15)
\end{aligned}
$$

Similarly one can estimate

$$
\begin{aligned}
&E\left(\int_{-\infty}^{a_i-a_{i+1}}\int_{y+a_i}^{y+a_{i+1}}L_t^xdx|y|^2 1_{\{|y|<1\}}n(dy)\right)^{\frac p2}\\
&\le c(\sigma,t,p)|a_{i+1}-a_i|^{\frac p2}\sup_x E(L_t^x)^{\frac p2}\left(\int_{-\infty}^{a_i-a_{i+1}}|y|^2 1_{\{|y|<1\}}n(dy)\right)^{\frac p2}. \qquad (3.16)
\end{aligned}
$$

Then we can estimate each term in (3.14). When $k=1$, we change the orders of the integration and use Jensen's inequality to have

$$
\begin{aligned}
&E\left(\int_{-\infty}^{a_i}L_t^x\int_{x-a_{i+1}}^{x-a_i}|x-y-a_i|^2 1_{\{|y|<1\}}n(dy)dx\right)^{\frac p2}\\
&= E\Bigg(\int_{-\infty}^{a_i-a_{i+1}}\int_{y+a_i}^{y+a_{i+1}}L_t^x(x-y-a_i)^2 1_{\{|y|<1\}}dxn(dy)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \int_{a_i-a_{i+1}}^{0}\int_{y+a_i}^{a_i} L_t^x (x-y-a_i)(x-y-a_i)1_{\{|y|<1\}}dxn(dy)\Bigg)^{\frac{p}{2}} \\
&\le E\Bigg(\int_{-\infty}^{a_i-a_{i+1}}\int_{y+a_i}^{y+a_{i+1}} L_t^x dx|y|^2 1_{\{|y|<1\}}n(dy) \\
&\quad + \int_{a_i-a_{i+1}}^{0}\int_{y+a_i}^{a_i} L_t^x (a_{i+1}-a_i)(x-y-a_i)dx1_{\{|y|<1\}}n(dy)\Bigg)^{\frac{p}{2}} \\
&\le c(\sigma,t,p)|a_{i+1}-a_i|^{\frac{p}{2}}.
\end{aligned}
\tag{3.17}
$$

For the term when $2 \le k \le m$, it is easy to see that

$$
\begin{aligned}
& E\left(\int_{-\infty}^{a_i} L_t^x \int_{x-a_{i+1}}^{x-a_i} |x-y-a_i|^{2^k} 1_{\{|y|<1\}}n(dy)dx\right)^{\frac{p}{2^k}} \\
&\le E\left(\int_{-\infty}^{a_i} L_t^x \int_{x-a_{i+1}}^{x-a_i} |x-y-a_i|^{2^{k-1}}|y|^{2^{k-1}} 1_{\{|y|<1\}}n(dy)dx\right)^{\frac{p}{2^k}} \\
&\le (a_{i+1}-a_i)^{\frac{1}{2}p} E\left(\int_{-\infty}^{a_i} L_t^x \int_{x-a_{i+1}}^{x-a_i} |y|^{2^{k-1}} 1_{\{|y|<1\}}n(dy)dx\right)^{\frac{p}{2^k}} \\
&\le c(t,p)|a_{i+1}-a_i|^{\frac{p}{2}}.
\end{aligned}
$$

Actually we can see that $\int_{-\infty}^{a_i}\int_{x-a_{i+1}}^{x-a_i}|y|^{2^{k-1}}1_{\{|y|<1\}}n(dy)dx < \infty$ by using the same method as in (3.17). For the last term in (3.14), similarly, we have

$$
\begin{aligned}
& \left(E\int_{-\infty}^{a_i} L_t^x \int_{x-a_{i+1}}^{x-a_i} |x-y-a_i|^{2^{m+1}} 1_{\{|y|<1\}}n(dy)dx\right)^{\frac{p}{2^{m+1}}} \\
&\le \left(E\int_{-\infty}^{a_i} L_t^x \int_{x-a_{i+1}}^{x-a_i} |a_{i+1}-a_i|^{2^m}|y|^{2^m} 1_{\{|y|<1\}}n(dy)dx\right)^{\frac{p}{2^{m+1}}} \\
&\le |a_{i+1}-a_i|^{\frac{p}{2}}\left(E\int_{-\infty}^{a_i} L_t^x \int_{x-a_{i+1}}^{x-a_i} |y|^{2^m} 1_{\{|y|<1\}}n(dy)dx\right)^{\frac{p}{2^{m+1}}} \\
&\le c(t,p)|a_{i+1}-a_i|^{\frac{p}{2}}.
\end{aligned}
$$

We can see that the key point is to estimate the term when $k=1$ because the higher order term can always be dealt by the above method easily. We can use the similar method to deal with other terms and derive that

$$
E|K_2(t,a_{i+1})-K_2(t,a_i)|^p \le c(t,\sigma,p)|a_{i+1}-a_i|^{\frac{p}{2}}. \tag{3.18}
$$

About $K_3(t,a)$, with the decomposition (3.5), we will estimate the sum of each term for jumps $|\Delta X_s| < 1$. There are six such terms.

For the first term in (3.5), by the p-moment estimate formula and occupation times formula, changing orders of integration and using Jensen's inequality, we have

$$
E\left(\int_0^t \int_{-\infty}^{\infty} |X_s-a_i|1_{\{X_s\le a_i\}}1_{\{a_i<X_s-y\le a_{i+1}\}}1_{\{|y|<1\}}n(dy)ds\right)^p
$$

$$
\begin{aligned}
&= \frac{1}{\sigma^{2p}} E\left(\int_{-\infty}^{a_i} L_t^x \int_{x-a_{i+1}}^{x-a_i} |x-a_i| 1_{\{|y|<1\}} n(dy) dx\right)^p \\
&= \frac{1}{\sigma^{2p}} E\Bigg(\int_{-\infty}^{a_i-a_{i+1}} \int_{y+a_i}^{y+a_{i+1}} L_t^x (a_i-x) 1_{\{|y|<1\}} dx n(dy) \\
&\qquad + \int_{a_i-a_{i+1}}^{0} \int_{y+a_i}^{a_i} L_t^x (a_i-x)^{\frac{1}{2}} (a_i-x)^{\frac{1}{2}} 1_{\{|y|<1\}} dx n(dy)\Bigg)^p \\
&\leq \frac{1}{\sigma^{2p}} E\Bigg(\int_{-\infty}^{a_i-a_{i+1}} \int_{y+a_i}^{y+a_{i+1}} L_t^x dx |y| 1_{\{|y|<1\}} n(dy) \\
&\qquad + \int_{a_i-a_{i+1}}^{0} \int_{y+a_i}^{a_i} L_t^x (a_{i+1}-a_i)^{\frac{1}{2}} (a_i-x)^{\frac{1}{2}} dx 1_{\{|y|<1\}} n(dy)\Bigg)^p \\
&\leq c(\sigma)|a_{i+1}-a_i|^{\frac{1}{2}p} \sup_x E(L_t^x)^p \left(\int_{-\infty}^{a_i-a_{i+1}} |y|^{\frac{3}{2}} 1_{\{|y|<1\}} n(dy)\right)^p \\
&\quad + c(\sigma)|a_{i+1}-a_i|^{\frac{1}{2}p} \sup_x E(L_t^x)^p \left(\int_{a_i-a_{i+1}}^{0} |y|^{\frac{3}{2}} 1_{\{|y|<1\}} n(dy)\right)^p \\
&\leq c(t,\sigma,p)|a_{i+1}-a_i|^{\frac{1}{2}p}.
\end{aligned}
\tag{3.19}
$$

We can use the similar method to deal with other terms. In the following, we will only sketch the estimate without giving great details.

2) For the second term, we have

$$
\begin{aligned}
&E\left(\int_0^t \int_{-\infty}^{\infty} (a_{i+1}-a_i) 1_{\{X_s \leq a_i\}} 1_{\{X_s - y > a_{i+1}\}} 1_{\{|y|<1\}} n(dy) ds\right)^p \\
&= \frac{1}{\sigma^{2p}} (a_{i+1}-a_i)^p E\left(\int_{-\infty}^{a_i} L_t^x \int_{-\infty}^{x-a_{i+1}} 1_{\{|y|<1\}} n(dy) dx\right)^p \\
&= c(\sigma)(a_{i+1}-a_i)^p E\left(\int_{-\infty}^{a_i-a_{i+1}} \left(\int_{y+a_{i+1}}^{a_i} L_t^x dx\right) 1_{\{|y|<1\}} n(dy)\right)^p \\
&\leq c(\sigma)|a_{i+1}-a_i|^{\frac{1}{2}p} \sup_x E(L_t^x)^p \left(\int_{-\infty}^{a_i-a_{i+1}} |y|^{\frac{3}{2}} 1_{\{|y|<1\}} n(dy)\right)^p \\
&\leq c(t,\sigma,p)|a_{i+1}-a_i|^{\frac{1}{2}p}.
\end{aligned}
$$

3) For the third term, we have

$$
\begin{aligned}
&E\left(\int_0^t \int_{-\infty}^{\infty} |X_s-a_i| 1_{\{a_i < X_s \leq a_{i+1}\}} 1_{\{X_s - y \leq a_i\}} 1_{\{|y|<1\}} n(dy) ds\right)^p \\
&= \frac{1}{\sigma^{2p}} E\left(\int_{a_i}^{a_{i+1}} L_t^x \int_{x-a_i}^{\infty} |x-a_i| 1_{\{|y|<1\}} n(dy) dx\right)^p \\
&\leq \frac{1}{\sigma^{2p}} E\Bigg(\int_0^{a_{i+1}-a_i} \int_{a_i}^{a_i+y} L_t^x (a_{i+1}-a_i)^{\frac{1}{2}} (x-a_i)^{\frac{1}{2}} dx 1_{\{|y|<1\}} n(dy) \\
&\quad + \int_{a_{i+1}-a_i}^{\infty} \int_{a_i}^{a_{i+1}} L_t^x dx |y| 1_{\{|y|<1\}} n(dy)\Bigg)^p \qquad (3.20) \\
&\leq c(\sigma,p)|a_{i+1}-a_i|^{\frac{1}{2}p} \sup_x E(L_t^x)^p \left(\int_0^{\infty} |y|^{\frac{3}{2}} 1_{\{|y|<1\}} n(dy)\right)^p
\end{aligned}
$$

$$\leq c(t,\sigma,p)|a_{i+1}-a_i|^{\frac{1}{2}p}.$$

The fourth, fifth and the last terms are symmetric to the third, second and first terms respectively. In summary, we have

$$E|K_3(t,a_{i+1})-K_3(t,a_i)|^p \leq c(t,\sigma,p)|a_{i+1}-a_i|^{\frac{1}{2}p}. \tag{3.21}$$

So we proved the result. $\diamond$

Remark 3.1 *From (3.19) and (3.20), we can see easily that if we require the following slightly stronger condition on the Lévy measure*

$$\int_{R\setminus\{0\}} (|y|^{\frac{3}{2}-\xi} \wedge 1)n(dy) < \infty, \tag{3.22}$$

for a $\xi \in (0,\frac{1}{2}]$*, then for any* $p \geq 1$*,*

$$E|K_3(t,a_{i+1})-K_3(t,a_i)|^p \leq c(t,\sigma,p)|a_{i+1}-a_i|^{(\frac{1}{2}+\xi)p}. \tag{3.23}$$

This estimate will be used in the construction of the geometric rough path where (3.21) is not adequate. In particular, this plays an essential role in obtaining (3.43) and (3.48), from which one can calculate $(Z(m)^2)_{m\in N}$ *is a Cauchy sequence in the* θ*-variation distance.*

Now, define $Z_x := (L_t^x, g(x))$ as a process of variable x in R^2. Here g is of bounded q-variation, $2 \leq q < 3$. Then it's easy to know that Z_x is of bounded $\hat{q}$-variation in x, where $\hat{q} = q$, if $q > 2$, and $\hat{q} > 2$ can be taken any number when $q = 2$. We assume a sightly stronger condition than (3.6) for the Lévy measure: there exists a constant $\varepsilon > 0$ such that

$$\int_{R\setminus\{0\}} (|y|^{1+\frac{1}{q}-(3-q)\varepsilon} \wedge 1)n(dy) < \infty. \tag{3.24}$$

We will prove with this condition, the desired geometric rough path $\mathbf{Z} = (1, \mathbf{Z}^1, \mathbf{Z}^2)$ is well defined. We need to point out that in the following when we consider the control function and the convergence of the first level path, condition (3.6) is still adequate. But we need (3.24) in the convergence of the second level path. Denote $\delta = \frac{1}{q} - (3-q)\varepsilon$. Note when $q = 2$, $\delta = \frac{1}{2} - \varepsilon$. So condition (3.24) becomes: there exists $\varepsilon > 0$ such that

$$\int_{R\setminus\{0\}} (|y|^{\frac{3}{2}-\varepsilon} \wedge 1)n(dy) < \infty. \tag{3.25}$$

Later in this section, we will see under this condition, the integral $\int_{-\infty}^{\infty} L_t^x dL_t^x$ can be well-defined as a rough path integral. Also note $\inf\limits_{2\leq q<3} \delta(q,\varepsilon) = \frac{1}{3}$. So under the condition

$$\int_{R\setminus\{0\}} (|y|^{\frac{4}{3}} \wedge 1)n(dy) < \infty, \tag{3.26}$$

(3.24) is satisfied for any $2 \leq q < 3$. In this case, our results imply that we can construct the geometric rough path for any g being of finite q-variation, where $2 \leq q < 3$ can be arbitrary.

Recall the θ-variation metric d_θ on $C_{0,\theta}(\varDelta, T^{([\theta])}(R^2))$ defined in [23],

$$d_\theta(\mathbf{Z},\mathbf{Y})=\max_{1\le i\le[\theta]}d_{i,\theta}(\mathbf{Z}^i,\mathbf{Y}^i)=\max_{1\le i\le[\theta]}\sup_D\left(\sum_l|\mathbf{Z}^i_{x_{l-1},x_l}-\mathbf{Y}^i_{x_{l-1},x_l}|^{\frac{\theta}{i}}\right)^{\frac{i}{\theta}}.$$

Assume condition (3.6) through to Proposition 3.1. Let $[x',x'']$ be any interval in R. From the proof of Theorem 2.1, for any $p\ge 2$, we know there exists a constant $c>0$ such that

$$E|L^b_t-L^a_t|^p\le c|b-a|^{\frac{p}{2}}, \tag{3.27}$$

i.e. L^x_t satisfies Hölder condition in [23] with exponent $\frac{1}{2}$. First we consider the case when g is continuous. Recall in [23], a control w is a continuous super-additive function on $\Delta:=\{(a,b):x'\le a<b\le x''\}$ with values in $[0,\infty)$ such that $w(a,a)=0$. Therefore

$$w(a,b)+w(b,c)\le w(a,c),\quad for\ any\ (a,b),(b,c)\in\Delta.$$

If $g(x)$ is of bounded q-variation, we can find a control w s.t.

$$|g(b)-g(a)|^q\le w(a,b),$$

for any $(a,b)\in\Delta:=\{(a,b):x'\le a<b\le x''\}$. It is obvious that $w_1(a,b):=w(a,b)+(b-a)$ is also a control of g. Set $h=\frac{1}{q}$, it is trivial to see for any $\theta>q$ (so $h\theta>1$) we have,

$$|g(b)-g(a)|^\theta\le w_1(a,b)^{h\theta},\ for\ any\ (a,b)\in\Delta. \tag{3.28}$$

Considering (3.27), we can see Z_x satisfies, for such $h=\frac{1}{q}$, and any $\theta>q$, there exists a constant c such that

$$E|Z_b-Z_a|^\theta\le cw_1(a,b)^{h\theta},\ for\ any\ (a,b)\in\Delta. \tag{3.29}$$

For any $m\in N$, define a continuous and bounded variation path $Z(m)$ by

$$Z(m)_x:=Z_{x^m_{l-1}}+\frac{w_1(x)-w_1(x^m_{l-1})}{w_1(x^m_l)-w_1(x^m_{l-1})}\Delta^m_l Z, \tag{3.30}$$

if $x^m_{l-1}\le x<x^m_l$, for $l=1,\cdots,2^m$, and $\Delta^m_l Z=Z_{x^m_l}-Z_{x^m_{l-1}}$. Here $D_m:=\{x'=x^m_0<x^m_1<\cdots<x^m_{2^m}=x''\}$ is a partition of $[x',x'']$ such that

$$w_1(x^m_l)-w_1(x^m_{l-1})=\frac{1}{2^m}w_1(x',x''),$$

where $w_1(x):=w_1(x',x)$. It is obvious that $x^m_l-x^m_{l-1}\le\frac{1}{2^m}w_1(x',x'')$ and by the superadditivity of the control function w_1,

$$w_1(x^m_{l-1},x^m_l)\le w_1(x^m_l)-w_1(x^m_{l-1})=\frac{1}{2^m}w_1(x',x'').$$

The corresponding smooth rough path $\mathbf{Z}(m)$ is built by taking its iterated path integrals, i.e. for any $(a,b)\in\Delta$,

$$\mathbf{Z}(m)^j_{a,b}=\int_{a<x_1<\cdots<x_j<b}dZ(m)_{x_1}\otimes\cdots\otimes dZ(m)_{x_j}. \tag{3.31}$$

In the following, we will prove $\{\mathbf{Z}(m)\}_{m\in N}$ converges to a geometric rough path $\mathbf{Z}$ in the θ-variation topology when $2\le q<3$. We call $\mathbf{Z}$ the canonical geometric rough path associated to Z.

Remark 3.2 *The bounded variation process $Z(m)_x$ is a generalized Wong-Zakai approximation to the process Z of bounded $\hat{q}$-variation. Here we divide $[x', x'']$ by equally partitioning the range of w_1. We then use (3.30) to form the piecewise curved approximation to Z. Note here Wong-Zakai's standard piecewise linear approximation does not work immediately.*

Remark 3.3 *It is noted here that there is no unique way to construct rough path. The approach we present in this paper is one construction that makes the Lévy area convergent. More importantly, we will see later the integral constructed coincides with the Lebesgue-Stieltjes integral when g is of bounded variation. And the convergence theorem of the integral (Propostion 4.2) guarantees that the integral we constructed in this paper is the limit of the Lebesgue-Stieltjes integrals in θ-variation topology. This shows that the rough path integral defined by (3.49) is the correct integral in our formula (4.7).*

Let's first look at the first level path $\mathbf{Z}(m)^1_{a,b}$. The method is similar to Chapter 4 in [23] for Brownian motion. Similar to Proposition 4.2.1 in [23], we can prove that for all $n \in N$, $m \mapsto \sum_{k=1}^{2^n} |\mathbf{Z}(m)^1_{x^n_{k-1},x^n_k}|^\theta$ is increasing and for $m \geq n$,

$$\mathbf{Z}(m)^1_{x^n_{k-1},x^n_k} = \mathbf{Z}(m+1)^1_{x^n_{k-1},x^n_k} = Z_{x^n_k} - Z_{x^n_{k-1}}. \tag{3.32}$$

Let $\mathbf{Z}^1_{a,b} = Z_b - Z_a$. Then (3.29) implies $E|\mathbf{Z}^1_{a,b}|^\theta \leq c w_1(a,b)^{h\theta}$. For such points $\{x^n_k\}$, $k = 1, \cdots, 2^n$, $n = 1, 2, \cdots$, defined above we still have the inequality in Proposition 4.1.1 in [23],

$$\begin{aligned} E \sup_D \sum_l |\mathbf{Z}^1_{x_{l-1},x_l}|^\theta &\leq C(\theta,\gamma) E \sum_{n=1}^\infty n^\gamma \sum_{k=1}^{2^n} |\mathbf{Z}^1_{x^n_{k-1},x^n_k}|^\theta \\ &\leq C_1 \sum_{n=1}^\infty n^\gamma (\frac{1}{2^n})^{h\theta-1} w_1(x',x'')^{h\theta}, \end{aligned} \tag{3.33}$$

for constant $C_1 = C(\theta,\gamma)c$. Since $h\theta - 1 > 0$, the series on the right-hand side of (3.33) is convergent, so that $\sup_D \sum_l |\mathbf{Z}^1_{x_{l-1},x_l}|^\theta < \infty$ almost surely. This shows that $\mathbf{Z}^1$ has finite θ-variation almost surely. Moreover, for any $\gamma > \theta - 1$, there exists a constant $C_1(\theta,\gamma,c) > 0$ such that

$$\begin{aligned} E \sup_m \sup_D \sum_l |\mathbf{Z}(m)^1_{x_{l-1},x_l}|^\theta &\leq C(\theta,\gamma) E \sup_m \sum_{n=1}^\infty n^\gamma \sum_{k=1}^{2^n} |\mathbf{Z}(m)^1_{x^n_{k-1},x^n_k}|^\theta \\ &\leq C(\theta,\gamma) E \sum_{n=1}^\infty n^\gamma \sum_{k=1}^{2^n} |\mathbf{Z}^1_{x^n_{k-1},x^n_k}|^\theta \\ &\leq C_1(\theta,\gamma,c) \sum_{n=1}^\infty n^\gamma (\frac{1}{2^n})^{h\theta-1} w_1(x',x'')^{h\theta} \\ &< \infty. \end{aligned} \tag{3.34}$$

So

$$\sup_m \sup_D \sum_l |\mathbf{Z}(m)^1_{x_{l-1},x_l}|^\theta < \infty \quad a.s. \tag{3.35}$$

This means that $\mathbf{Z}(m)^1_{a,b}$ have finite θ-variation uniformly in m. And furthermore, from (3.32) and some standard arguments,

$$E\sum_{n=1}^{\infty} n^{\gamma} \sum_{k=1}^{2^n} |\mathbf{Z}(m)^1_{x^n_{k-1},x^n_k} - \mathbf{Z}^1_{x^n_{k-1},x^n_k}|^{\theta} \le C(\frac{1}{2^m})^{\frac{h\theta-1}{2}}, \tag{3.36}$$

where C depends on θ, h, $w_1(x', x'')$, and c in (3.29). By Proposition 4.1.2 in [23], Jensen's inequality and (3.36),

$$\begin{aligned} E\sum_{m=1}^{\infty} \sup_D \Big(\sum_l |\mathbf{Z}(m)^1_{x_{l-1},x_l} - \mathbf{Z}^1_{x_{l-1},x_l}|^{\theta} \Big)^{\frac{1}{\theta}} &\le E\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} n^{\gamma} \sum_{k=1}^{2^n} |\mathbf{Z}(m)^1_{x^n_{k-1},x^n_k} - \mathbf{Z}^1_{x^n_{k-1},x^n_k}|^{\theta} \right)^{\frac{1}{\theta}} \\ &\le \sum_{m=1}^{\infty} \left(E\sum_{n=1}^{\infty} n^{\gamma} \sum_{k=1}^{2^n} |\mathbf{Z}(m)^1_{x^n_{k-1},x^n_k} - \mathbf{Z}^1_{x^n_{k-1},x^n_k}|^{\theta} \right)^{\frac{1}{\theta}} \\ &\le C\sum_{m=1}^{\infty} (\frac{1}{2^m})^{\frac{h\theta-1}{2\theta}} \\ &< \infty, \end{aligned} \tag{3.37}$$

for $h\theta > 1$. So we obtain

Theorem 3.1 *Let L^x_t be the local time of the time homogeneous Lévy process X_t given by (1.1), and g be a continuous function of bounded q-variation. Assume $q \ge 1$, $\sigma \ne 0$ and the Lévy measure $n(dy)$ satisfies (3.6). Then for any $\theta > q$, the continuous process $Z_x = (L^x_t, g(x))$ satisfying (3.29), we have*

$$\sum_{m=1}^{\infty} \sup_D \Big(\sum_l |\mathbf{Z}(m)^1_{x_{l-1},x_l} - \mathbf{Z}^1_{x_{l-1},x_l}|^{\theta} \Big)^{\frac{1}{\theta}} < \infty \ a.s. \tag{3.38}$$

In particular, $(\mathbf{Z}(m)^1_{a,b})$ converges to $(\mathbf{Z}^1_{a,b})$ in the θ-variation distance a.s. for any $(a, b) \in \Delta$.

We next consider the second level path $\mathbf{Z}(m)^2_{a,b}$. As in [23], we can also see that if $m \le n$,

$$\mathbf{Z}(m)^2_{x^n_{k-1},x^n_k} = 2^{2(m-n)-1}(\Delta^m_l Z)^{\otimes 2}, \tag{3.39}$$

where l is chosen such that $x^m_{l-1} \le x^n_{k-1} < x^n_k \le x^m_l$; if $m > n$,

$$\mathbf{Z}(m)^2_{x^n_{k-1},x^n_k} = \frac{1}{2}\Delta^n_k Z \otimes \Delta^n_k Z + \frac{1}{2} \sum_{\substack{r,l=2^{m-n}(k-1)+1 \\ r<l}}^{2^{m-n}k} (\Delta^m_r Z \otimes \Delta^m_l Z - \Delta^m_l Z \otimes \Delta^m_r Z),$$

so

$$\begin{aligned} &\mathbf{Z}(m+1)^2_{x^n_{k-1},x^n_k} - \mathbf{Z}(m)^2_{x^n_{k-1},x^n_k} \\ =& \frac{1}{2} \sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k} (\Delta^{m+1}_{2l-1} Z \otimes \Delta^{m+1}_{2l} Z - \Delta^{m+1}_{2l} Z \otimes \Delta^{m+1}_{2l-1} Z), \end{aligned} \tag{3.40}$$

$k = 1, \cdots, 2^n$. Similar to the proof of Proposition 4.3.3 in [23], we have

Proposition 3.1 *Assume g is a continuous function of finite q-variation with a real number $q \ge 2$, and the Lévy measure satisfies (3.6). Let $\theta > q$. Then for $m \le n$,*

$$\sum_{k=1}^{2^n} E|\mathbf{Z}(m+1)^2_{x^n_{k-1},x^n_k} - \mathbf{Z}(m)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}} \le C(\frac{1}{2^{n+m}})^{\frac{\theta h-1}{2}}, \tag{3.41}$$

where C depends on θ, $h(:= \frac{1}{q})$, $w_1(x', x'')$, and c in (3.29).

The main step to establish the geometric rough path integral over Z is the following estimate. Delicate and correct power in (3.43) must be obtained to prove the convergence of the approximated Lévy area. We will use Lemma 3.3 about the correlation of $K_2(t,a_{i+1})-K_2(t,a_i)$ and $K_2(t,a_{j+1})-K_2(t,a_j)$, and (3.23) for the term K_3, for $\xi=\frac{q-2}{2q}+(3-q)\varepsilon$.

Lemma 3.3 *Assume the Lévy measure satisfies (3.22) with* $0<\xi\leq\frac{1}{6}$, *then for any* $a_0<a_1<\cdots<a_m$,

$$\Big|E\big(K_2(t,a_{i+1})-K_2(t,a_i)\big)\big(K_2(t,a_{j+1})-K_2(t,a_j)\big)\Big|$$
$$\leq\begin{cases} c(t,\sigma)(a_{i+1}-a_i), & \text{when } 0\leq i=j\leq m,\\ c(t,\sigma)[(a_{i+1}-a_i)^{1+2\xi}+(a_{j+1}-a_j)^{1+2\xi}], & \text{when } 0\leq i\neq j\leq m.\end{cases} \tag{3.42}$$

Proof: When $i=j$, (3.42) follows from (3.18) directly. Here we only need Lévy measure satisfies (2.2). Now we consider the case when $i\neq j$. Without losing generality, we assume $i<j$. From (3.4), it is easy to see that

$$\begin{aligned}
&E\big(K_2(t,a_{i+1})-K_2(t,a_i)\big)\cdot\big(K_2(t,a_{j+1})-K_2(t,a_j)\big)\\
&=E\left[\int_0^{t+}\int_R J_*(X_s,X_s-y,a_i,a_{i+1})\tilde N_p(dyds)\cdot\int_0^{t+}\int_R J_*(X_s,X_s-y,a_j,a_{j+1})\tilde N_p(dyds)\right]\\
&=E\int_0^t\int_R J_*(X_s,X_s-y,a_i,a_{i+1})J_*(X_s,X_s-y,a_j,a_{j+1})n(dy)ds\\
&=A_1+A_2+A_3+A_4,
\end{aligned}$$

where

$$\begin{aligned}
A_1:=&\int_0^t\int_R\Big[-(a_{i+1}-a_i)(a_j-X_s+y)\Big]1_{\{X_s\leq a_i\}}1_{\{a_j<X_s-y\leq a_{j+1}\}}1_{\{|y|<1\}}n(dy)ds\\
&+\int_0^t\int_R\Big[(a_{i+1}-a_i)(a_{j+1}-a_j)\Big]1_{\{X_s\leq a_i\}}1_{\{X_s-y>a_{j+1}\}}1_{\{|y|<1\}}n(dy)ds,\\
A_2:=&\int_0^t\int_R\Big[-(a_{i+1}-X_s)(a_j-X_s+y)\Big]1_{\{a_i<X_s\leq a_{i+1}\}}1_{\{a_j<X_s-y\leq a_{j+1}\}}1_{\{|y|<1\}}n(dy)ds\\
&+\int_0^t\int_R(a_{i+1}-X_s)(a_{j+1}-a_j)1_{\{a_i<X_s\leq a_{i+1}\}}1_{\{X_s-y>a_{j+1}\}}1_{\{|y|<1\}}n(dy)ds,\\
A_3:=&\int_0^t\int_R\Big[(a_{i+1}-a_i)(X_s-a_j)\Big]1_{\{X_s-y\leq a_i\}}1_{\{a_j<X_s\leq a_{j+1}\}}1_{\{|y|<1\}}n(dy)ds\\
&+\int_0^t\int_R(a_{i+1}-a_i)(a_{j+1}-a_j)1_{\{X_s-y\leq a_i\}}1_{\{X_s>a_{j+1}\}}1_{\{|y|<1\}}n(dy)ds,\\
A_4:=&\int_0^t\int_R\Big[-(X_s-y-a_{i+1})(X_s-a_j)1_{\{a_i<X_s-y\leq a_{i+1}\}}1_{\{a_j<X_s\leq a_{j+1}\}}1_{\{|y|<1\}}n(dy)ds\\
&+\int_0^t\int_R\Big[-(X_s-y-a_{i+1})(a_{j+1}-a_j)\Big]1_{\{a_i<X_s-y\leq a_{i+1}\}}1_{\{X_s>a_{j+1}\}}\Big]1_{\{|y|<1\}}n(dy)ds.
\end{aligned}$$

For convenience, we denote the two integrals in A_1 by A_{11} and A_{12} respectively. To estimate $|EA_1|$, first by the occupation times formula, Fubini theorem, Jensen's inequality, similar as before, we have :

$$|EA_{11}|\leq\frac{1}{\sigma^2}E\int_{-\infty}^{a_i}L_t^x\int_{x-a_{j+1}}^{x-a_j}(a_{i+1}-a_i)(x-a_j-y)1_{\{|y|<1\}}n(dy)dx$$

$$
\begin{aligned}
&\le \frac{1}{\sigma^2}(a_{i+1}-a_i)^{\frac12+\xi}E\Big[\int_{-\infty}^{a_i-a_{j+1}}|y|^{\frac12-\xi}\int_{y+a_j}^{y+a_{j+1}}L_t^x|y|1_{\{|y|<1\}}dxn(dy)\\
&\qquad+\int_{a_i-a_{j+1}}^{a_i-a_j}|y|^{\frac12-\xi}\int_{y+a_j}^{a_i}L_t^x|y|1_{\{|y|<1\}}dxn(dy)\Big]\\
&\le \frac{1}{\sigma^2}(a_{i+1}-a_i)^{\frac12+\xi}(a_{j+1}-a_j)(\sup_x EL_t^x)\int_{-\infty}^{a_i-a_j}|y|^{\frac32-\xi}1_{\{|y|<1\}}n(dy)\\
&\le c(t,\sigma)(a_{i+1}-a_i)^{\frac12+\xi}(a_{j+1}-a_j).
\end{aligned}
$$

In the same way, we can have

$$
\begin{aligned}
|EA_{12}| &\le \frac{1}{\sigma^2}(a_{i+1}-a_i)^{\frac12+\xi}(a_{j+1}-a_j)\sup_x E(L_t^x)\int_{-\infty}^{a_i-a_{j+1}}|y|^{\frac32-\xi}|y|1_{\{|y|<1\}}n(dy)\\
&\le c(t,\sigma)(a_{i+1}-a_i)^{\frac12+\xi}(a_{j+1}-a_j).
\end{aligned}
$$

Therefore, we get

$$
|EA_1| \le c(t,\sigma)((a_{i+1}-a_i)^{1+2\xi}+(a_{j+1}-a_j)^{1+2\xi}).
$$

Using the same method, we can have the similar estimations for $|EA_2|$, $|EA_3|$ and $|EA_4|$. It follows that when $i \neq j$,

$$
|E\big(K_2(t,a_{i+1})-K_2(t,a_i)\big)\big(K_2(t,a_{j+1})-K_2(t,a_j)\big)| \le c(t,\sigma)((a_{i+1}-a_i)^{1+2\xi}+(a_{j+1}-a_j)^{1+2\xi}).
$$

Therefore we proved (3.42). ◇

Proposition 3.2 *Assume g is a continuous function of finite q-variation with a real number $q \in [2,3)$, and the Lévy measure satisfies (3.24). Let $q<\theta<3$. Then for $m>n$, we have*

$$
E|\mathbf{Z}(m+1)^2_{x^n_{k-1},x^n_k}-\mathbf{Z}(m)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}} \le C\Big[(\frac{1}{2^n})^{\frac{\theta}{4}}(\frac{1}{2^m})^{\frac12 h\theta}+(\frac{1}{2^n})^{\frac{\theta}{2}}(\frac{1}{2^m})^{\frac{3-q}{2}\varepsilon\theta}\Big], \tag{3.43}
$$

where C is a generic constant and also depends on θ, $h(:=\frac1q)$, $w_1(x',x'')$, and c in (3.29).

Proof: For $m>n$, it is easy to see that

$$
\begin{aligned}
&E|\mathbf{Z}(m+1)^2_{x^n_{k-1},x^n_k}-\mathbf{Z}(m)^2_{x^n_{k-1},x^n_k}|^2\\
&=\frac14 E\Big|\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k}(\Delta^{m+1}_{2l-1}Z\otimes\Delta^{m+1}_{2l}Z-\Delta^{m+1}_{2l}Z\otimes\Delta^{m+1}_{2l-1}Z)\Big|^2\\
&=\frac14\sum_{\substack{i,j=1\\ i\neq j}}^{2}E\sum_{l,r=2^{m-n}(k-1)+1}^{2^{m-n}k}(\Delta^{m+1}_{2l-1}Z^i\Delta^{m+1}_{2l}Z^j-\Delta^{m+1}_{2l}Z^i\Delta^{m+1}_{2l-1}Z^j)\\
&\qquad\qquad\cdot(\Delta^{m+1}_{2r-1}Z^i\Delta^{m+1}_{2r}Z^j-\Delta^{m+1}_{2r}Z^i\Delta^{m+1}_{2r-1}Z^j)\\
&=\frac14\sum_{l,r}\Big[E(\Delta^{m+1}_{2l-1}L^x_t\Delta^{m+1}_{2r-1}L^x_t)(\Delta^{m+1}_{2l}g(x)\Delta^{m+1}_{2r}g(x))\\
&\qquad+(\Delta^{m+1}_{2l-1}g(x)\Delta^{m+1}_{2r-1}g(x))E(\Delta^{m+1}_{2l}L^x_t\Delta^{m+1}_{2r}L^x_t)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{4}\sum_{l,r}\Big[E(\Delta^{m+1}_{2l-1}L^x_t\Delta^{m+1}_{2r}L^x_t)(\Delta^{m+1}_{2l}g(x)\Delta^{m+1}_{2r-1}g(x))\\
&\qquad+(\Delta^{m+1}_{2l-1}g(x)\Delta^{m+1}_{2r}g(x))E(\Delta^{m+1}_{2l}L^x_t\Delta^{m+1}_{2r-1}L^x_t)\Big]\\
&-\frac{1}{4}\sum_{l,r}\Big[E(\Delta^{m+1}_{2l}L^x_t\Delta^{m+1}_{2r-1}L^x_t)(\Delta^{m+1}_{2l-1}g(x)\Delta^{m+1}_{2r}g(x))\\
&\qquad+(\Delta^{m+1}_{2l}g(x)\Delta^{m+1}_{2r-1}g(x))E(\Delta^{m+1}_{2l-1}L^x_t\Delta^{m+1}_{2r}L^x_t)\Big]\\
&+\frac{1}{4}\sum_{l,r}\Big[E(\Delta^{m+1}_{2l}L^x_t\Delta^{m+1}_{2r}L^x_t)(\Delta^{m+1}_{2l-1}g(x)\Delta^{m+1}_{2r-1}g(x))\\
&\qquad+(\Delta^{m+1}_{2l}g(x)\Delta^{m+1}_{2r}g(x))E(\Delta^{m+1}_{2l-1}L^x_t\Delta^{m+1}_{2r-1}L^x_t)\Big].
\end{aligned}
\tag{3.44}
$$

The main difficulty is to estimate the following expectation which can be derived from Tanaka's formula:

$$
\begin{aligned}
&E\Big[\Delta^{m+1}_{2r-1}L^x_t\Delta^{m+1}_{2l-1}L^x_t\Big]\\
=&E\Big[\big(L_t(x^{m+1}_{2r-1})-L_t(x^{m+1}_{2r-2})\big)\big(L_t(x^{m+1}_{2l-1})-L_t(x^{m+1}_{2l-2})\big)\Big]\\
=&E\Big[\varphi_t(x^{m+1}_{2r-1})-\varphi_t(x^{m+1}_{2r-2})-b\int_0^t 1_{\{x^{m+1}_{2r-2}<X_{s-}\le x^{m+1}_{2r-1}\}}ds\\
&\quad-\sigma\int_0^t 1_{\{x^{m+1}_{2r-2}<X_{s-}\le x^{m+1}_{2r-1}\}}dB_s+(K_1(t,x^{m+1}_{2r-1})-K_1(t,x^{m+1}_{2r-2}))\\
&\quad+(K_2(t,x^{m+1}_{2r-1})-K_2(t,x^{m+1}_{2r-2}))+(K_3(t,x^{m+1}_{2r-1})-K_3(t,x^{m+1}_{2r-2}))\Big]\\
&\cdot\Big[\varphi_t(x^{m+1}_{2l-1})-\varphi_t(x^{m+1}_{2l-2})-b\int_0^t 1_{\{x^{m+1}_{2l-2}<X_{s-}\le x^{m+1}_{2l-1}\}}ds\\
&\quad-\sigma\int_0^t 1_{\{x^{m+1}_{2l-2}<X_{s-}\le x^{m+1}_{2l-1}\}}dB_s+(K_1(t,x^{m+1}_{2l-1})-K_1(t,x^{m+1}_{2l-2}))\\
&\quad+(K_2(t,x^{m+1}_{2l-1})-K_2(t,x^{m+1}_{2l-2}))+(K_3(t,x^{m+1}_{2l-1})-K_3(t,x^{m+1}_{2l-2}))\Big].
\end{aligned}
$$

Firstly, from (3.8), (3.11), (3.12), the Cauchy-Schwarz inequality and the quadratic variation of stochastic integrals, we have

$$
\begin{aligned}
&\Big|E\big(\varphi_t(x^{m+1}_{2r-1})-\varphi_t(x^{m+1}_{2r-2})-b\int_0^t 1_{\{x^{m+1}_{2r-2}<X_{s-}\le x^{m+1}_{2r-1}\}}ds-\sigma\int_0^t 1_{\{x^{m+1}_{2r-2}<X_{s-}\le x^{m+1}_{2r-1}\}}dB_s\big)\\
&\quad\cdot\big(\varphi_t(x^{m+1}_{2l-1})-\varphi_t(x^{m+1}_{2l-2})-b\int_0^t 1_{\{x^{m+1}_{2l-2}<X_{s-}\le x^{m+1}_{2l-1}\}}ds-\sigma\int_0^t 1_{\{x^{m+1}_{2l-2}<X_{s-}\le x^{m+1}_{2l-1}\}}dB_s\big)\Big|\\
\le& C\Big[(1+2b+b^2)(x^{m+1}_{2r-1}-x^{m+1}_{2r-2})(x^{m+1}_{2l-1}-x^{m+1}_{2l-2})+\sigma(x^{m+1}_{2r-1}-x^{m+1}_{2r-2})(x^{m+1}_{2l-1}-x^{m+1}_{2l-2})^{\frac{1}{2}}\\
&\quad+\sigma(x^{m+1}_{2l-1}-x^{m+1}_{2l-2})(x^{m+1}_{2r-1}-x^{m+1}_{2r-2})^{\frac{1}{2}}\Big]\\
&+\sigma^2E\Big|\int_0^t 1_{\{x^{m+1}_{2r-2}\le X_s<x^{m+1}_{2r-1}\}}1_{\{x^{m+1}_{2l-2}\le X_s<x^{m+1}_{2l-1}\}}ds\Big|\\
\le& C\Big[(\frac{1}{2^{m+1}})^2w_1(x',x'')^2+(\frac{1}{2^{m+1}})^{\frac{3}{2}}w_1(x',x'')^{\frac{3}{2}}\Big]\\
&+\sigma^2E\Big|\int_0^t 1_{\{x^{m+1}_{2r-2}\le X_s<x^{m+1}_{2r-1}\}}1_{\{x^{m+1}_{2l-2}\le X_s<x^{m+1}_{2l-1}\}}ds\Big|\\
\le&\begin{cases} C(\frac{1}{2^{m+1}})^{\frac{3}{2}}, & \text{if } r\ne l,\\ C\frac{1}{2^{m+1}}, & \text{if } r=l.\end{cases}
\end{aligned}
\tag{3.45}
$$

Here C is a generic constant and may depend on t, b, σ, $w_1(x', x'')$. Secondly, recall the fact that $E(P_{\cdot}M_{\cdot}) = 0$, if P is a process of bounded variation and M is a martingale with mean 0 and at least one of M and P is continuous. Note here K_1 is a process of bounded variation. Recall also that the cross-variation of $\int_0^t 1_{\{a_i < X_{s-} \le a_{i+1}\}} dB_s$ and the jump parts such as $\big(K_2(t, a_{j+1}) - K_2(t, a_j)\big)$ are zero. So we have

$$\begin{aligned} &E\int_0^t 1_{\{a_i<X_{s-}\le a_{i+1}\}}dB_s\cdot\big(K_d(t,a_{j+1})-K_d(t,a_j)\big)=0,\ \ d=1,2,3,\\ &E\int_0^t 1_{\{a_i<X_{s-}\le a_{i+1}\}}ds\cdot\big(K_2(t,a_{j+1})-K_2(t,a_j)\big)=0,\\ &E\big(K_2(t,a_{i+1})-K_2(t,a_i)\big)\cdot\big(K_3(t,a_{j+1})-K_3(t,a_j)\big)=0. \end{aligned}$$

Thirdly, by Lemma 3.3, we can see that when $0 \le \xi \le \frac{1}{6}$, $i \neq j$

$$\Big|E\big(K_2(t,a_{i+1})-K_2(t,a_i)\big)\big(K_2(t,a_{j+1})-K_2(t,a_j)\big)\Big|\le C[(a_{i+1}-a_i)^{1+2\xi}+(a_{j+1}-a_j)^{1+2\xi}].$$

For other terms, by the Cauchy-Schwarz inequality, (3.23) and (3.42), it is easy to see that

$$\begin{aligned} &\Big|E\big(\varphi_t(a_{i+1})-\varphi_t(a_i)\big)\big(K_d(t,a_{j+1})-K_d(t,a_j)\big)\Big|\le C[(a_{i+1}-a_i)(a_{j+1}-a_j)^{\alpha_d}],d=1,2,3;\\ &\Big|E\int_0^t 1_{\{a_i<X_{s-}\le a_{i+1}\}}ds\big(K_d(t,a_{j+1})-K_d(t,a_j)\big)\Big|\le C[(a_{i+1}-a_i)(a_{j+1}-a_j)^{\alpha_d}],d=1,3;\\ &\Big|E\big(K_1(t,a_{i+1})-K_1(t,a_i)\big)\big(K_d(t,a_{j+1})-K_d(t,a_j)\big)\Big|\le C[(a_{i+1}-a_i)(a_{j+1}-a_j)^{\alpha_d}],d=1,2,3;\\ &\Big|E\big(K_3(t,a_{i+1})-K_3(t,a_i)\big)\big(K_3(t,a_{j+1})-K_3(t,a_j)\big)\Big|\le C[(a_{i+1}-a_i)^{\frac12+\xi}(a_{j+1}-a_j)^{\frac12+\xi}]. \end{aligned}$$

Here $\alpha_1 = 1$, $\alpha_2 = \frac{1}{2}$ and $\alpha_3 = \frac{1}{2} + \xi$. Thus

$$\Big|E\Big[\Delta_{2r-1}^{m+1}L_t^x\Delta_{2l-1}^{m+1}L_t^x\Big]\Big|\le\begin{cases}C(\frac{1}{2^{m+1}})^{1+2\xi}, & \text{if } r\neq l,\\ C\frac{1}{2^{m+1}}, & \text{if } r=l.\end{cases}\tag{3.46}$$

The other terms in (3.44) can be treated similarly. Therefore

$$E|\mathbf{Z}(m+1)^2_{x^n_{k-1},x^n_k}-\mathbf{Z}(m)^2_{x^n_{k-1},x^n_k}|^2\le C\Big[2^{m-n}(\frac{1}{2^{m+1}})^{1+2h}+2^{2(m-n)}(\frac{1}{2^{m+1}})^{1+2\xi+2h}\Big].$$

Hence, for $2 < \theta < 3$, by Jensen's inequality,

$$\begin{aligned} E|\mathbf{Z}(m+1)^2_{x^n_{k-1},x^n_k}-\mathbf{Z}(m)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}}&\le\Big(E|\mathbf{Z}(m+1)^2_{x^n_{k-1},x^n_k}-\mathbf{Z}(m)^2_{x^n_{k-1},x^n_k}|^2\Big)^{\frac{\theta}{4}}\\ &\le C\Big[2^{m-n}(\frac{1}{2^{m+1}})^{1+2h}+2^{2(m-n)}(\frac{1}{2^{m+1}})^{1+2\xi+2h}\Big]^{\frac{\theta}{4}}\\ &\le C\Big[2^{(m-n)\frac{\theta}{4}}(\frac{1}{2^{m+1}})^{\frac{\theta}{4}+\frac12 h\theta}+2^{(m-n)\frac{\theta}{2}}(\frac{1}{2^{m+1}})^{\frac{1+2\xi}{4}\theta+\frac12 h\theta}\Big]\\ &\le C\Big[(\frac{1}{2^n})^{\frac{\theta}{4}}(\frac{1}{2^m})^{\frac12 h\theta}+(\frac{1}{2^n})^{\frac{\theta}{2}}(\frac{1}{2^m})^{\frac12 h\theta-\frac{1-2\xi}{4}\theta}\Big], \end{aligned}$$

where C is a generic constant and also depends on θ, h, $w_1(x', x'')$, and c. Note $\xi = \frac{q-2}{2q} + (3-q)\varepsilon$, so we get (3.43). ◇

Corollary 3.1 *Under the same assumption as in Proposition 3.2, we have*

$$\sup_m\sum_{n=1}^\infty n^\gamma\sum_{k=1}^{2^n}|\mathbf{Z}(m)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}}<\infty\quad a.s.$$

Proof: From the Minkowski inequality,

$$\begin{aligned}
&\left(\sum_{k=1}^{2^n}|\mathbf{Z}(m)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}\\
\le&\left(\sum_{k=1}^{2^n}|\mathbf{Z}(m)^2_{x^n_{k-1},x^n_k}-\mathbf{Z}(m-1)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}+\left(\sum_{k=1}^{2^n}|\mathbf{Z}(m-1)^2_{x^n_{k-1},x^n_k}-\mathbf{Z}(m-2)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}\\
&+\cdots+\left(\sum_{k=1}^{2^n}|\mathbf{Z}(1)^2_{x^n_{k-1},x^n_k}-\mathbf{Z}(0)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}+\left(\sum_{k=1}^{2^n}|\mathbf{Z}(0)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}\\
\le&\sum_{m=1}^{\infty}\left(\sum_{k=1}^{2^n}|\mathbf{Z}(m)^2_{x^n_{k-1},x^n_k}-\mathbf{Z}(m-1)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}+\left(\sum_{k=1}^{2^n}|\mathbf{Z}(0)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}. \qquad (3.47)
\end{aligned}$$

Then it is easy to see from (3.47), Jensen's inequality, (3.41), (3.43) and (3.39)

$$\begin{aligned}
&E\sup_m\sum_{n=1}^{\infty}n^{\frac{2}{\theta}\gamma}\left(\sum_{k=1}^{2^n}|\mathbf{Z}(m)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}\\
\le&E\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}n^{\frac{2}{\theta}\gamma}\left(\sum_{k=1}^{2^n}|\mathbf{Z}(m)^2_{x^n_{k-1},x^n_k}-\mathbf{Z}(m-1)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}\right.\\
&\qquad\left.+\sum_{n=1}^{\infty}n^{\frac{2}{\theta}\gamma}E\left(\sum_{k=1}^{2^n}|\mathbf{Z}(0)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}\right]\\
\le&\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}n^{\frac{2}{\theta}\gamma}\left(E\sum_{k=1}^{2^n}|\mathbf{Z}(m)^2_{x^n_{k-1},x^n_k}-\mathbf{Z}(m-1)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}\right.\\
&\qquad\left.+\sum_{n=1}^{\infty}n^{\frac{2}{\theta}\gamma}(2^{n+\frac{(-1-2n)\theta}{2}}w(x',x'')^{h\theta})^{\frac{2}{\theta}}\right]\\
\le&C\sum_{n=m}^{\infty}n^{\frac{2}{\theta}\gamma}\sum_{m=1}^{\infty}(\frac{1}{2^{n+m}})^{(\frac{h\theta-1}{2})\frac{2}{\theta}}\\
&+C\sum_{n=1}^{\infty}n^{\frac{2}{\theta}\gamma}\sum_{m=1}^{\infty}\left[(\frac{1}{2^n})^{(\frac{\theta}{4}-\frac{1}{2})\frac{2}{\theta}}(\frac{1}{2^m})^{(\frac{1}{2}h\theta-\frac{1}{2})\frac{2}{\theta}}+(\frac{1}{2^n})^{1-\frac{2}{\theta}}(\frac{1}{2^m})^{(\frac{3-q}{2}\varepsilon\theta)\frac{2}{\theta}}\right]\\
&+C\sum_{n=1}^{\infty}n^{\frac{2}{\theta}\gamma}(\frac{1}{2^n})^{\frac{2(\theta-1)}{\theta}}\\
<&\infty,
\end{aligned}$$

as $2<\theta<3$, $h\theta>1$, where C is a generic constant and also depends on θ, h, $w_1(x',x'')$, and c. Therefore,

$$\sup_m\sum_{n=1}^{\infty}n^{\frac{\theta}{2}\gamma}\left(\sum_{k=1}^{2^n}|\mathbf{Z}(m)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}<\infty \quad a.s.$$

However, it is easy to see as $\theta>2$,

$$\left(\sup_m\sum_{n=1}^{\infty}n^{\gamma}\sum_{k=1}^{2^n}|\mathbf{Z}(m)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}\le\sup_m\sum_{n=1}^{\infty}n^{\frac{2}{\theta}\gamma}\left(\sum_{k=1}^{2^n}|\mathbf{Z}(m)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}<\infty \quad a.s.$$

So the claim follows. $\diamond$

Theorem 3.2 *Let L_t^x be the local time of the time homogeneous Lévy process X_t given by (1.1). Assume g is a continuous function of finite q-variation with a real number $2 \leq q < 3$, and the Lévy measure $n(dy)$ satisfies (3.24), $\sigma \neq 0$. Then for any $\theta \in (q,3)$, the continuous process $Z_x = (L_t^x, g(x))$ satisfying (3.29), there exists a unique $\mathbf{Z}^i$ on Δ taking values in $(R^2)^{\otimes i}$ $(i = 1,2)$ such that*

$$\sum_{i=1}^{2} \sup_D \left(\sum_l |\mathbf{Z}(m)^i_{x_{l-1},x_l} - \mathbf{Z}^i_{x_{l-1},x_l}|^{\frac{\theta}{i}} \right)^{\frac{i}{\theta}} \to 0,$$

both almost surely and in $L^1(\Omega, \mathcal{F}, \mathcal{P})$ as $m \to \infty$. In particular, $\mathbf{Z} = (1, \mathbf{Z}^1, \mathbf{Z}^2)$ is the canonical geometric rough path associated to $Z_\cdot$, and $\mathbf{Z}^1_{a,b} = Z_b - Z_a$.

Proof: The convergence of $\mathbf{Z}(m)^1$ to $\mathbf{Z}^1$ is actually the result of Theorem 3.1. In the following we will prove $\mathbf{Z}(m)^2_{a,b}$ converges in the θ-variation distance. By Proposition 4.1.2 in [23],

$$\begin{aligned}
& E \sup_D \sum_l |\mathbf{Z}(m+1)^2_{x_{l-1},x_l} - \mathbf{Z}(m)^2_{x_{l-1},x_l}|^{\frac{\theta}{2}} \\
\leq\ & C(\theta,\gamma) E \left(\sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} |\mathbf{Z}(m+1)^1_{x^n_{k-1},x^n_k} - \mathbf{Z}(m)^1_{x^n_{k-1},x^n_k}|^\theta \right)^{\frac{1}{2}} \\
& \cdot \left(\sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} \left(|\mathbf{Z}(m+1)^1_{x^n_{k-1},x^n_k}|^\theta + |\mathbf{Z}(m)^1_{x^n_{k-1},x^n_k}|^\theta \right) \right)^{\frac{1}{2}} \\
& + C(\theta,\gamma) E \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} |\mathbf{Z}(m+1)^2_{x^n_{k-1},x^n_k} - \mathbf{Z}(m)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}} \\
:=\ & A + B.
\end{aligned}$$

We will estimate part A, B respectively. First from the Cauchy-Schwarz inequality, (3.34) and (3.36), we know

$$\begin{aligned}
A \leq\ & C \Bigg(E \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} \left(|\mathbf{Z}(m+1)^1_{x^n_{k-1},x^n_k} - \mathbf{Z}^1_{x^n_{k-1},x^n_k}|^\theta + |\mathbf{Z}(m)^1_{x^n_{k-1},x^n_k} - \mathbf{Z}^1_{x^n_{k-1},x^n_k}|^\theta \right) \Bigg)^{\frac{1}{2}} \\
& \cdot \Bigg(E \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} \left(|\mathbf{Z}(m+1)^1_{x^n_{k-1},x^n_k}|^\theta + |\mathbf{Z}(m)^1_{x^n_{k-1},x^n_k}|^\theta \right) \Bigg)^{\frac{1}{2}} \\
\leq\ & C (\frac{1}{2^m})^{\frac{h\theta-1}{4}} \left(\sum_{n=1}^{\infty} n^\gamma (\frac{1}{2^n})^{h\theta-1} \right)^{\frac{1}{2}}.
\end{aligned}$$

Secondly from Proposition 3.1 and Proposition 3.2, we know

$$\begin{aligned}
B \leq\ & C \sum_{n=m}^{\infty} n^\gamma (\frac{1}{2^{m+n}})^{\frac{h\theta-1}{2}} + C \Bigg[\sum_{n=1}^{m-1} n^\gamma (\frac{1}{2^n})^{\frac{\theta}{4}-1} (\frac{1}{2^m})^{\frac{h\theta}{2}} + \sum_{n=1}^{m-1} n^\gamma (\frac{1}{2^n})^{\frac{\theta}{2}-1} (\frac{1}{2^m})^{\frac{3-q}{2}\varepsilon\theta} \Bigg] \\
\leq\ & C \Bigg[(\frac{1}{2^m})^{\frac{h\theta-1}{2}} + (\frac{1}{2^m})^{\frac{h\theta-1}{2}} + (\frac{1}{2^m})^{\frac{3-q}{2}\varepsilon\theta} \Bigg],
\end{aligned}$$

as $q < \theta < 3$, and $h\theta > 1$. So

$$E\sup_D\sum_l|\mathbf{Z}(m+1)^2_{x_{l-1},x_l}-\mathbf{Z}(m)^2_{x_{l-1},x_l}|^{\frac{\theta}{2}}\leq C[(\frac{1}{2^m})^{\frac{h\theta-1}{4}}+(\frac{1}{2^m})^{\frac{3-q}{2}\varepsilon\theta}]. \tag{3.48}$$

Similar to the proof of Theorem 3.1, we can easily deduce that $(\mathbf{Z}(m)^2)_{m\in N}$ is a Cauchy sequence in the θ-variation distance. So when $m\to\infty$, it has a limit, denote it by $\mathbf{Z}^2$. And from the completeness under the θ-variation distance (Lemma 3.3.3 in [23]), $\mathbf{Z}^2$ is also of finite θ-variation. The theorem is proved. ⋄

Remark 3.4 *We would like to point out that the above method does not seem to work for two arbitrary functions f of p-variation and g of q-variation ($2<p,q<3$) to define a rough path $Z_x=(f(x),g(x))$. However the special property (3.46) of local times makes our analysis work. A similar method was used in [23] for fractional Brownian motion with the help of long-time memory. Here (3.46) serves a similar role of the long-time memory as in [23].*

Remark 3.5 *The requirement (3.24) on Lévy measure can guarantee us to prove (3.48) so that we can obtain the desired convergence. We think if we relax the condition on Lévy measure, we probably need to calculate the higher level of rough path. We don't include this analysis in here as it is already a long paper.*

As local time L_t^x has a compact support in x for each ω and t, so we can define integral of local time directly in R. For this, we take $[x',x'']$ covering the support of L_t^x. From Chen's identity, it's easy to know that for any $(a,b)\in\Delta$,

$$\mathbf{Z}^2_{a,b}=\lim_{m(D_{[a,b]})\to 0}\sum_{i=0}^{r-1}(\mathbf{Z}^2_{x_i,x_{i+1}}+\mathbf{Z}^1_{a,x_i}\otimes\mathbf{Z}^1_{x_i,x_{i+1}}).$$

In particular,

$$\begin{aligned}(\mathbf{Z}^2_{a,b})_{2,1}&=\lim_{m(D_{[a,b]})\to 0}\sum_{i=0}^{r-1}((\mathbf{Z}^2_{x_i,x_{i+1}})_{2,1}+(\mathbf{Z}^1_{a,x_i}\otimes\mathbf{Z}^1_{x_i,x_{i+1}})_{2,1})\\&=\lim_{m(D_{[a,b]})\to 0}\sum_{i=0}^{r-1}((\mathbf{Z}^2_{x_i,x_{i+1}})_{2,1}+(g(x_i)-g(a))(L_t^{x_{i+1}}-L_t^{x_i}))\end{aligned}$$

exists. Here $(\mathbf{Z}^2_{x_i,x_{i+1}})_{2,1}$ means lower-left element of the 2×2 matrix $\mathbf{Z}^2_{x_i,x_{i+1}}$. It turns out that

$$\begin{aligned}&\lim_{m(D_{[a,b]})\to 0}\sum_{i=0}^{r-1}((\mathbf{Z}^2_{x_i,x_{i+1}})_{2,1}+g(x_i)(L_t^{x_{i+1}}-L_t^{x_i}))\\=&\lim_{m(D_{[a,b]})\to 0}\sum_{i=0}^{r-1}((\mathbf{Z}^2_{x_i,x_{i+1}})_{2,1}+(g(x_i)-g(a))(L_t^{x_{i+1}}-L_t^{x_i}))+g(a)(L_t^b-L_t^a)\end{aligned}$$

exists. Denote this limit by $\int_a^b g(x)dL_t^x$. Similarly, we can define $\int_a^b L_t^x dL_t^x$. To verify the latter integral is well defined, we only need to consider the case $q=2$. Then it is easy to see under condition (3.25), $\int_a^b L_t^x dL_t^x$ is defined as a rough path integral. Therefore we have the following corollary.

Corollary 3.2 *Assume all conditions of Theorem 3.2, but the Lévy measure satisfies (3.25). Then the local time L_t^x is a geometrical rough path of roughness p in x for any $t\geq 0$ a.s. for any $p>2$, and $(a,b)\in\Delta$,*

$$\int_a^b L_t^x dL_t^x = \lim_{m(D_{[a,b]})\to 0} \sum_{i=0}^{r-1} ((\mathbf{Z}^2_{x_i,x_{i+1}})_{1,1} + L(x_i)(L_t^{x_{i+1}} - L_t^{x_i})).$$

Moreover, if g is a continuous function with bounded q-variation, $2 \le q < 3$, and the Lévy measure satisfies (3.24), then the integral $\int_a^b g(x)dL_t^x$ is defined by

$$\int_a^b g(x)dL_t(x) = \lim_{m(D_{[a,b]})\to 0} \sum_{i=0}^{r-1} ((\mathbf{Z}^2_{x_i,x_{i+1}})_{2,1} + g(x_i)(L_t^{x_{i+1}} - L_t^{x_i})). \tag{3.49}$$

4 Continuity of the rough path integrals and applications to extensions of Itô's formula

In this section we will apply the Young integral and rough path integral of local time defined in sections 2 and 3 to prove a useful extension to Itô's formula. First we consider some convergence result of the rough path integrals.

Let $Z_j(x) := (L_t^x, g_j(x))$, where $g_j(\cdot)$ is of bounded q-variation uniformly in j for $2 \le q < 3$, and when $j \to \infty$, $g_j(x) \to g(x)$ for all $x \in R$. Assume the Lévy measure satisfies (3.24). Repeating the above argument, for each j, we can find the canonical geometric rough path $\mathbf{Z}_j = (1, \mathbf{Z}_j^1, \mathbf{Z}_j^2)$ associated to Z_j, and the smooth rough path $\mathbf{Z}_j(m) = (1, \mathbf{Z}_j(m)^1, \mathbf{Z}_j(m)^2)$. Actually, $(\mathbf{Z}_j)^1_{a,b} \to \mathbf{Z}^1_{a,b}$ in the sense of the uniform topology, and also in the sense of the θ-variation topology. As for $(\mathbf{Z}_j)^2_{a,b}$, we can easily see that

$$d_{2,\theta}((\mathbf{Z}_j)^2, \mathbf{Z}^2) \le d_{2,\theta}((\mathbf{Z}_j)^2, (\mathbf{Z}_j(m))^2) + d_{2,\theta}((\mathbf{Z}_j(m))^2, \mathbf{Z}(m)^2) + d_{2,\theta}(\mathbf{Z}(m)^2, \mathbf{Z}^2). \tag{4.1}$$

From Theorem 3.2, we know that $d_{2,\theta}(\mathbf{Z}(m)^2, \mathbf{Z}^2) \to 0$ as $m \to \infty$. Moreover, it is not difficult to see from the proofs of Propositions 3.1, 3.2, and Theorem 3.2, $d_{2,\theta}((\mathbf{Z}_j)^2, (\mathbf{Z}_j(m))^2) \to 0$ as $m \to \infty$ uniformly in j. So for any given $\varepsilon > 0$, there exists an m_0 such that when $m \ge m_0$, $d_{2,\theta}(\mathbf{Z}(m)^2, \mathbf{Z}^2) < \frac{\varepsilon}{3}$, $d_{2,\theta}((\mathbf{Z}_j)^2, (\mathbf{Z}_j(m))^2) < \frac{\varepsilon}{3}$ for all j. In particular, $d_{2,\theta}(\mathbf{Z}(m_0)^2, \mathbf{Z}^2) < \frac{\varepsilon}{3}$, $d_{2,\theta}((\mathbf{Z}_j)^2, (\mathbf{Z}_j(m_0))^2) < \frac{\varepsilon}{3}$ for all j. It's easy to prove for such m_0, $d_{2,\theta}((\mathbf{Z}_j(m_0))^2, \mathbf{Z}(m_0)^2) < \frac{\varepsilon}{3}$ for sufficiently large j. Replacing m by m_0 in (4.1), we can get $d_{2,\theta}((\mathbf{Z}_j)^2, \mathbf{Z}^2) < \varepsilon$ for sufficiently large j. Then by (3.49) and the definition of $\int_a^b g_j(x)dL_t^x$, we know that $\int_a^b g_j(x)dL_t^x \to \int_a^b g(x)dL_t^x$ as $j \to \infty$. Similarly, we can see from the last section, when we consider $\mathbf{Z}_t(m) = (1, \mathbf{Z}_t(m)^1, \mathbf{Z}_t(m)^2)$, $d_{2,\theta}((\mathbf{Z}_t)^2, (\mathbf{Z}_t(m))^2) \to 0$, as $m \to \infty$ uniformly in $t \in [0,T]$, for any $T > 0$. Therefore we can also conclude that $\mathbf{Z}_t^2$ is continuous in t in the $d_{2,\theta}$ topology. Note now that the local time L_t^x has a compact support in x a.s. So it is easy to see from taking $[x', x'']$ covering the support of L_t^x that the above construction of the integrals and the convergence can work for the integrals on R. Therefore we have

Proposition 4.1 *Let $Z_j(x) := (L_t^x, g_j(x))$, $Z(x) := (L_t^x, g(x))$, where $g_j(\cdot)$, $g(\cdot)$ are continuous and of bounded q-variation uniformly in j, $2 \le q < 3$, and the Lévy measure $n(dy)$ satisfies (3.24), $\sigma \neq 0$. Assume $g_j(x) \to g(x)$ as $j \to \infty$ for all $x \in R$. Then as $j \to \infty$, $\mathbf{Z}_j(\cdot) \to \mathbf{Z}(\cdot)$ a.s. in the θ-variation distance. In particular, as $j \to \infty$, $\int_{-\infty}^{\infty} g_j(x)dL_t^x \to \int_{-\infty}^{\infty} g(x)dL_t^x$a.s. Similarly, $Z_t(\cdot)$ is continuous in t in the θ-topology. In particular, $\int_{-\infty}^{\infty} g(x)dL_t^x$ is a continuous function of t a.s.*

Now for any g being continuous and of bounded q-variation ($2 \le q < 3$), define

$$g_j(x) = \int_{-\infty}^{\infty} k^j(x-y)g(y)dy, \tag{4.2}$$

where k^j is the mollifier given by

$$k^j(x) = \begin{cases} cj\mathrm{e}^{\frac{1}{(jx-1)^2-1}}, & \text{if } x \in (0, \frac{2}{j}), \\ 0, & \text{otherwise.} \end{cases}$$

Here c is a constant such that $\int_0^2 k^j(x)dx = 1$. It is well known that g_j is a smooth function and $g_j(x) \to g(x)$ as $j \to \infty$ for each x. So the integral $\int_{-\infty}^{\infty} g_j(x)dL_t^x$ is a Riemann integral for the smooth function $g_j(x)$. Moreover, Proposition 4.1 guarantees that $\int_{-\infty}^{\infty} g_j(x)dL_t^x \to \int_{-\infty}^{\infty} g(x)dL_t^x$ a.s.

In the following, we will show that Proposition 4.1 is true for g being of bounded q-variation ($2 \le q < 3$) without assuming g being continuous. Note that a function with bounded q-variation ($q \ge 1$) may have at most countable discontinuities. Using the method in [35], we will define the rough path integral $\int_{x'}^{x''} L_t^x dg(x)$. Here we assume $g(x)$ is càdlàg in x.

First we can define a map

$$\tau_\delta(\cdot) : [x', x''] \to [x', x'' + \delta \sum_{n=1}^{\infty} |j(x_n)|^q],$$

in the following way:

$$\tau_\delta(x) = x + \delta \sum_{n=1}^{\infty} |j(x_n)|^q 1_{\{x_n \le x\}}(x),$$

where $j(x_i) := G(x_i) - G(x_i-)$, $\{x_i\}_{i=1}^{\infty}$ are the discontinuous points of G inside $[x', x'']$, $\delta > 0$. The map $\tau_\delta(\cdot) : [x', x''] \to [x', \tau_\delta(x'')]$ extends the space interval into the one where we can define the continuous path $G_\delta(y)$ from a càdlàg path G by:

$$G_\delta(y) = \begin{cases} G(x) & \text{if } y = \tau_\delta(x), \\ G(x_n-) + (y - \tau_\delta(x_n-))j(x_n)\delta^{-1}|j(x_n)|^{-q} & \text{if } y \in [\tau_\delta(x_n-), \tau_\delta(x_n)). \end{cases} \tag{4.3}$$

Take G to be g and L_t, we can define g_δ and $L_{t,\delta}$ respectively. As L_t^x is continuous, we can easily see that $L_{t,\delta}(y) := L_{t,\delta}(\tau_\delta(x)) = L_t^x$.

Theorem 4.1 *Let $g(x)$ be a càdlàg path with bounded q-variation ($2 \le q < 3$), and the Lévy measure $n(dy)$ satisfies (3.24), $\sigma \neq 0$. Then*

$$\int_{x'}^{x''} L_t^x dg(x) = \int_{x'}^{\tau_\delta(x'')} L_{t,\delta}(y)dg_\delta(y). \tag{4.4}$$

Proof: First it is easy to see that the integral $\int_{x'}^{\tau_\delta(x'')} L_{t,\delta}(y)dg_\delta(y)$ is a rough path integral that can be defined by the method of last section. Now note that at any discontinuous point x_r,

$$\int_{x_r-}^{x_r} L_t^x dg(x) = L_t(x_r)(g(x_r) - g(x_r-))$$

and

$$\sum_r ((Z_\delta)^2_{\tau_\delta(x_r-),\tau_\delta(x_r)})_{2,1} = \sum_r \int_{\tau_\delta(x_r-)}^{\tau_\delta(x_r)} (L_{t,\delta}(y) - L_{t,\delta}(\tau_\delta(x_r-)))dg_\delta(y) = 0,$$

where $Z_\delta(y) := (L_{t,\delta}(y), g_\delta(y))$. Thus

$$
\begin{aligned}
&\sum_r L_t^{x_r}(g(x_r) - g(x_r-)) \\
&= \sum_r L_{t,\delta}(\tau_\delta(x_r-))(g_\delta(\tau_\delta(x_r)) - g_\delta(\tau_\delta(x_r-))) \\
&= \sum_r \left[L_{t,\delta}(\tau_\delta(x_r-))(g_\delta(\tau_\delta(x_r)) - g_\delta(\tau_\delta(x_r-))) + ((Z_\delta)^2_{\tau_\delta(x_r-),\tau_\delta(x_r)})_{2,1} \right] \\
&< \infty,
\end{aligned}
$$

so

$$
\begin{aligned}
\int_{\tau_\delta(x_r-)}^{\tau_\delta(x_r)} L_{t,\delta}(y) dg_\delta(y) &= L_{t,\delta}(\tau_\delta(x_r-))(g_\delta(\tau_\delta(x_r)) - g_\delta(\tau_\delta(x_r-))) \\
&= L_t(x_r)(g(x_r) - g(x_{r-})).
\end{aligned}
$$

Thus

$$
\int_{x_r-}^{x_r} L_t^x dg(x) = \int_{\tau_\delta(x_r-)}^{\tau_\delta(x_r)} L_{t,\delta}(y) dg_\delta(y).
$$

Now define $g(x) = \tilde{g}(x) + h(x)$, where $h(x) = \sum_{x_r \le x} (g(x_r) - g(x_{r-}))$. Then $\tilde{g}$ is the continuous part of g and h is the jump part of g. Moreover, $\tilde{g}$ satisfies the q-variation condition. So $\int_{x'}^{x''} L_t(x) d\tilde{g}(x)$ can be well defined as in the last section. For h, we can define h_δ by taking $G = h$ in (4.3). So the integral $\int_{x'}^{x"} L_t^x dh(x)$ can be well defined by the followings:

$$
\begin{aligned}
\int_{x'}^{\tau_\delta(x'')} L_{t,\delta}(y) dh_\delta(y) &= \sum_r \int_{\tau_\delta(x_{r-})}^{\tau_\delta(x_r)} L_{t,\delta}(y) dh_\delta(y) = \sum_r L_t(x_r)(h(x_r) - h(x_{r-})) \\
&= \sum_r L_t(x_r)(g(x_r) - g(x_{r-})) = \sum_r \int_{x_{r-}}^{x_r} L_t^x dh(x) = \int_{x'}^{x''} L_t^x dh(x).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int_{x'}^{x''} L_t^x dg(x) &= \int_{x'}^{x''} L_t^x d\tilde{g}(x) + \int_{x'}^{x''} L_t^x dh(x) \\
&= \int_{x'}^{\tau_\delta(x'')} L_{t,\delta}(y) d\tilde{g}_\delta(y) + \int_{x'}^{\tau_\delta(x'')} L_{t,\delta}(y) dh_\delta(y) \\
&= \int_{x'}^{\tau_\delta(x'')} L_{t,\delta}(y) dg_\delta(y).
\end{aligned}
$$

$\diamond$

Similarly to Proposition 4.1, we have

Proposition 4.2 *Under the condition of Proposition 4.1, as $j \to \infty$, $\int_{-\infty}^{\infty} L_t^x dg_j(x) \to \int_{-\infty}^{\infty} L_t^x dg(x)$ a.s. for such g with bounded q-variation ($2 \le q < 3$).*

Proof: Define $F_j(x) := (g_j - g)(x)$, so $F_j(x) \to 0$ as $j \to \infty$, for all x. It's easy to see that $F_{j,\delta}(x) \to 0$ as $j \to \infty$, for all x. From the above theorem and Proposition 4.1, we have

$$
\int_{x'}^{x''} L_t^x d(g_j - g)(x) = \int_{x'}^{x''} L_t^x dF_j(x) = \int_{x'}^{\tau_\delta(x'')} L_{t,\delta} dF_{j,\delta}(y) \to 0, \quad as\ j \to \infty.
$$

Then the proposition follows easily. ◇

Applying the standard smoothing procedure on $f(x)$, we can get $f_n(x)$ which is defined in the same way as $g_j(x)$ in (4.2). And by Itô's formula (c.f. [29]), we have

$$f_n(X_t) = f_n(X_0) + \int_0^t f_n'(X_s)dX_s + A_t^n, \quad 0 \le t < \infty. \tag{4.5}$$

where

$$A_t^n = \frac{1}{2}\int_0^t f_n''(X_{s-})d[X,X]_s^c + \sum_{0\le s\le t}[f_n(X_s) - f_n(X_{s-}) - f_n'(X_{s-})\Delta X_s]. \tag{4.6}$$

From the occupation times formula, the definition of the integral of local time and the convergence results of the integrals, we have

$$\lim_{n\to\infty}\frac{1}{2}\int_0^t f_n''(X_{s-})d[X,X]_s^c = \lim_{n\to\infty}\frac{1}{2}\int_{-\infty}^{\infty} L_t^x df_n'(x) = -\frac{1}{2}\int_{-\infty}^{\infty} f_-'(x)d_xL_t^x,$$

and the rough path integral $\int_{-\infty}^{\infty} f_-'(x)d_xL_t^x$ is continuous in t from Proposition 4.1. For the convergence of jump part in (4.6), we can conclude from the proof of Theorem 3 in Eisenbaum and Kyprianou [7], if the following assumption:

Condition (A): $\int_{\{|y|<1\}}|f(x+y) - f(x) - f_-'(x)y|n(dy)$ *is well defined and locally bounded in* x, holds, then

$$\lim_{n\to\infty}\sum_{0\le s\le t}[f_n(X_s) - f_n(X_{s-}) - f_n'(X_{s-})\Delta X_s] = \sum_{0\le s\le t}[f(X_s) - f(X_{s-}) - \nabla^- f(X_{s-})\Delta X_s],$$

in $L^2(dP)$. Therefore we have:

Theorem 4.2 *Let* $f: R \to R$ *be an absolutely continuous function and have left derivative* $f_-'(x)$ *being left continuous and locally bounded,* $f_-'(x)$ *be of bounded q-variation, where* $1 \le q < 3$. *Then for* $X = (X_t)_{t\ge 0}$, *a time homogeneous Lévy process with* $\sigma \ne 0$ *and Lévy measure* $n(dy)$ *satisfying Condition (A), and (3.6) when* $1 \le q < 2$, *(3.24) when* $2 \le q < 3$, *we have P-a.s.*

$$\begin{aligned} f(X_t) = f(X_0) &+ \int_0^t f_-'(X_s)dX_s - \frac{1}{2}\int_{-\infty}^{\infty} f_-'(x)d_xL_t^x \\ &+ \sum_{0\le s\le t}[f(X_s) - f(X_{s-}) - f_-'(X_{s-})\Delta X_s], \quad 0 \le t < \infty. \end{aligned} \tag{4.7}$$

Here the integral $\int_{-\infty}^{\infty} f_-'(x)d_xL_t^x$ *is a Lebesgue-Stieltjes integral when* $q = 1$, *a Young integral when* $1 < q < 2$ *and a Lyons' rough path integral when* $2 \le q < 3$ *respectively. In particular, under the condition (3.26), (4.7) holds for any* $2 \le q < 3$.

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