

# Stationary Solutions of SPDEs and Infinite Horizon BDSDEs with Non-Lipschitz Coefficients

Qi Zhang <sup>a,b,c</sup>, Huaizhong Zhao <sup>a</sup>

<sup>a</sup> Department of Mathematical Sciences, Loughborough University, Loughborough, LE11 3TU, UK.

<sup>b</sup> School of Mathematics and System Sciences, Shandong University, Jinan, 250100, China.

<sup>c</sup> Current address: School of Mathematical Sciences, Fudan University, Shanghai, 200433, China.

Emails: qzh@fudan.edu.cn; H.Zhao@lboro.ac.uk

**Summary.** We prove a general theorem that the  $L^2_\rho(\mathbb{R}^d; \mathbb{R}^1) \otimes L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)$  valued solution of an infinite horizon backward doubly stochastic differential equation, if exists, gives the stationary solution of the corresponding stochastic partial differential equation. We prove the existence and uniqueness of the  $L^2_\rho(\mathbb{R}^d; \mathbb{R}^1) \otimes L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)$  valued solutions for backward doubly stochastic differential equations on finite and infinite horizon with linear growth without assuming Lipschitz conditions, but under the monotonicity condition. Therefore the solution of finite horizon problem gives the solution of the initial value problem of the corresponding stochastic partial differential equations, and the solution of the infinite horizon problem gives the stationary solution of the SPDEs according to our general result.

**Keywords:** backward doubly stochastic differential equations, weak solutions, stochastic partial differential equations, pathwise stationary solution, monotone coefficients.

AMS 2000 subject classifications: 60H15, 60H10, 37H10.

## 1 Introduction, Basic Notation and Main Results

The notion of pathwise stationary solutions for stochastic partial differential equations (SPDEs) is a fundamental concept in the study of the long time behaviour of stochastic dynamical systems driven by SPDEs. It describes the pathwise invariance of the stationary solution, over time, along a measurable and  $P$ -preserving transformation  $\theta_t : \Omega \rightarrow \Omega$  and the pathwise limit of solutions of the random dynamical systems. It is a random fixed point  $Y(\omega)$  in the state space of the random dynamical system, in the sense that the solution  $v(t, Y(\omega), \omega)$  of the SPDE with initial value  $Y(\omega)$  is equal to  $Y(\theta_t \omega)$ , which is still  $Y$ , but with a different sample path  $\theta_t \omega$ . Therefore  $Y(\theta_t \omega)$  is a particular solution of the SPDE with the pathwise stationary property. Needless to say that the “one-force, one-solution” setting is a natural extension of the equilibrium or fixed point in the theory of the deterministic dynamical systems to stochastic counterparts. Such a random fixed point consists of infinitely many randomly moving invariant surfaces on the configuration space due to the random external force pumped to the system constantly. The study of its existence and stability is of great interests in both mathematics and physics. We would like to point out that the existence of stationary solutions is a basic assumption in many works on random dynamical systems e.g. in the study of stability (Has’minskii [11]), and in the theory of stable and unstable manifolds (Arnold [1], Mohammed, Zhang and Zhao [17], Duan, Lu and Schmalfuss [9]). But these theories give neither the existence of stationary solutions, nor a way of finding them. However, in contrast to the deterministic dynamical systems, the existence of stationary solutions of random dynamical systems is a more difficult and subtle problem. It is easy to see that the solutions of elliptic type partial differential equations give the stationary

solutions of the corresponding parabolic type partial differential equations, though the elliptic partial differential equations are difficult problems to study as well. However, for stochastic partial differential equations of the parabolic type, such kind of connection does not exist. In [17], Mohammed, Zhang and Zhao introduced an integral equation of infinite horizon for the stationary solutions of certain stochastic evolution equations. But the existence of solutions of such stochastic integral equations in general is far from clear. In [25], Zhang and Zhao proved that the solution of an infinite horizon backward doubly stochastic differential equation (BDSDE) under Lipschitz condition, if exists, is a perfect stationary solution. Moreover, under the Lipschitz and monotone conditions, the solution indeed exists and gives the stationary solution of the corresponding SPDEs of the parabolic type. It was known that the solutions of infinite horizon backward stochastic differential equations (BSDEs) give a classical or viscosity solution of elliptic type partial differential equations (Poisson equations) from the works of Peng [21] and Pardoux [18]. So philosophically, it is very natural to represent the stationary solutions of SPDEs as solutions of the corresponding infinite horizon BDSDEs, like the case of the Poisson equations as the solutions of the infinite horizon backward stochastic differential equations. Other works on stationary solutions of certain types of SPDEs usually under additive or linear noise include Sinai [23], [24], Caraballo, Kloeden, Schmalfuss [7].

In this paper, we will put above idea on infinite horizon BDSDEs in a general setting and prove a general theorem which basically says, if the infinite horizon BDSDE has a unique solution in the space  $S^{2,-K} \cap M^{2,-K}([0, \infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,-K}([0, \infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^d))$  for a  $K > 0$ , and the finite horizon BDSDE gives the representation for the solution of the corresponding SPDE, then, the solution of the infinite horizon BDSDE gives the stationary solution of the corresponding SPDE. Following this result, to study the existence of stationary solutions of SPDEs is transformed to study the existence and uniqueness of the solutions of the corresponding infinite horizon BDSDEs. In [25], we studied such equations when the nonlinear coefficients are assumed to be Lipschitz continuous. In this paper, we continue our work [25] to study the weak solution (in the weighted Sobolev space  $H^1_\rho(\mathbb{R}^d; \mathbb{R}^1)$  space) of the following parabolic SPDE without assuming the Lipschitz continuity of  $f$  on  $v$ :

$$\begin{aligned} dv(t, x) = & [\mathcal{L}v(t, x) + f(x, v(t, x), \sigma^*(x)Dv(t, x))]dt \\ & + g(x, v(t, x), \sigma^*(x)Dv(t, x))dB_t. \end{aligned} \quad (1.1)$$

Here  $B$  is a two-sided cylindrical Brownian motion valued on a separable Hilbert space  $U_0$  in a probability space  $(\Omega, \mathcal{F}, P)$ ;  $\mathcal{L}$  is the infinitesimal generator of a diffusion process  $X_s^{t,x}$  (the solution of Eq.(1.5)) given by

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i} \quad (1.2)$$

with  $(a_{ij}(x)) = \sigma \sigma^*(x)$ ;  $L^2_\rho(\mathbb{R}^d; \mathbb{R}^1)$  is the Hilbert space with the inner product

$$\langle u_1, u_2 \rangle = \int_{\mathbb{R}^d} u_1(x) u_2(x) \rho^{-1}(x) dx,$$

i.e. a  $\rho$ -weighted  $L^2$  space, where  $\rho(x) = (1 + |x|)^q : \mathbb{R}^d \longrightarrow \mathbb{R}^1$ ,  $q > d + 2$ , is a weight function. It is easy to see that  $\rho(x)$  is a continuous positive function satisfying  $\int_{\mathbb{R}^d} |x|^p \rho^{-1}(x) dx < \infty$  for any  $p \in (2, q - d)$ . Note that we can consider more general  $\rho$  which satisfies the above condition and conditions in [3] and all the results of this paper still hold. The SPDEs we study

in this paper are very general with the noise term  $g$  being allowed to be nonlinear in  $v$  and  $\nabla v$ . However, many techniques of [25] when  $f$  is Lipschitz do not work here. Although we can follow a similar procedure, as in [25], to consider first the finite horizon BDSDEs, then to make the connection with the weak solutions of the corresponding SPDEs and to find a Cauchy sequence to pass the terminal time of BDSDEs to infinity, we have to introduce new techniques to deal with the difficulties arising from the lack of the Lipschitz continuity of  $f$  on  $y$ .

Define  $u(t, x) = v(T - t, x)$  for arbitrary  $T$  and  $0 \leq t \leq T$ . We can show that  $u$  satisfies the following backward SPDE:

$$\begin{cases} du(t, x) + [\mathcal{L}u(t, x) + f(x, u(t, x), (\sigma^* \nabla u)(t, x))]dt - g(x, u(t, x), (\sigma^* \nabla u)(t, x))d^\dagger \hat{B}_t = 0 \\ u(T, x) = v(0, x). \end{cases} \quad (1.3)$$

Here  $\mathcal{L}$  is given by (1.2) and  $\hat{B}_s = B_{T-s} - B_T$ . Let  $\mathcal{N}$  denote the class of  $P$ -null sets of  $\mathcal{F}$ . We define

$$\begin{aligned} \mathcal{F}_{t,T} &\triangleq \mathcal{F}_{t,T}^{\hat{B}} \bigvee \mathcal{F}_t^W \bigvee \mathcal{N}, \quad \text{for } 0 \leq t \leq T; \\ \mathcal{F}_t &\triangleq \mathcal{F}_{t,\infty}^{\hat{B}} \bigvee \mathcal{F}_t^W \bigvee \mathcal{N}, \quad \text{for } t \geq 0. \end{aligned}$$

Recall Definitions 2.1 and 2.2 in [25]:

**Definition 1.1** Let  $\mathbb{S}$  be a Hilbert space with norm  $\|\cdot\|_{\mathbb{S}}$  and Borel  $\sigma$ -field  $\mathcal{S}$ . For  $K \in \mathbb{R}^+$ , we denote by  $M^{2,-K}([0, \infty); \mathbb{S})$  the set of  $\mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{F}/\mathcal{S}$  measurable random processes  $\{\phi(s)\}_{s \geq 0}$  with values in  $\mathbb{S}$  satisfying

- (i)  $\phi(s) : \Omega \rightarrow \mathbb{S}$  is  $\mathcal{F}_s$  measurable for  $s \geq 0$ ;
- (ii)  $E[\int_0^\infty e^{-Ks} \|\phi(s)\|_{\mathbb{S}}^2 ds] < \infty$ .

Also we denote by  $S^{2,-K}([0, \infty); \mathbb{S})$  the set of  $\mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{F}/\mathcal{S}$  measurable random processes  $\{\psi(s)\}_{s \geq 0}$  with values in  $\mathbb{S}$  satisfying

- (i)  $\psi(s) : \Omega \rightarrow \mathbb{S}$  is  $\mathcal{F}_s$  measurable for  $s \geq 0$  and  $\psi(\cdot, \omega)$  is continuous  $P$ -a.s.;
- (ii)  $E[\sup_{s \geq 0} e^{-Ks} \|\psi(s)\|_{\mathbb{S}}^2] < \infty$ .

**Definition 1.2** Let  $\mathbb{S}$  be a Hilbert space with norm  $\|\cdot\|_{\mathbb{S}}$  and Borel  $\sigma$ -field  $\mathcal{S}$ . We denote by  $M^{2,0}([t, T]; \mathbb{S})$  the set of  $\mathcal{B}_{[t,T]} \otimes \mathcal{F}/\mathcal{S}$  measurable random processes  $\{\phi(s)\}_{t \leq s \leq T}$  with values in  $\mathbb{S}$  satisfying

- (i)  $\phi(s) : \Omega \rightarrow \mathbb{S}$  is  $\mathcal{F}_{s,T} \bigvee \mathcal{F}_{T,\infty}^{\hat{B}}$  measurable for  $t \leq s \leq T$ ;
- (ii)  $E[\int_t^T \|\phi(s)\|_{\mathbb{S}}^2 ds] < \infty$ .

Also we denote by  $S^{2,0}([t, T]; \mathbb{S})$  the set of  $\mathcal{B}_{[t,T]} \otimes \mathcal{F}/\mathcal{S}$  measurable random processes  $\{\psi(s)\}_{t \leq s \leq T}$  with values in  $\mathbb{S}$  satisfying

- (i)  $\psi(s) : \Omega \rightarrow \mathbb{S}$  is  $\mathcal{F}_{s,T} \bigvee \mathcal{F}_{T,\infty}^{\hat{B}}$  measurable for  $t \leq s \leq T$  and  $\psi(\cdot, \omega)$  is continuous  $P$ -a.s.;
- (ii)  $E[\sup_{t \leq s \leq T} \|\psi(s)\|_{\mathbb{S}}^2] < \infty$ .

Recall also the weak solution of the SPDE (1.3) as follows for the convenience of the reader.

**Definition 1.3** A process  $u$  is called a weak solution (solution in  $L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)$ ) of Eq.(1.3) if  $(u, \sigma^* \nabla u) \in M^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d, \mathbb{R}^1)) \otimes M^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$  and for an arbitrary  $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^1)$ ,

$$\begin{aligned}
& \int_{\mathbb{R}^d} u(t, x) \varphi(x) dx - \int_{\mathbb{R}^d} u(T, x) \varphi(x) dx - \frac{1}{2} \int_t^T \int_{\mathbb{R}^d} (\sigma^* \nabla u)(s, x) (\sigma^* \nabla \varphi)(x) dx ds \\
& - \int_t^T \int_{\mathbb{R}^d} u(s, x) \operatorname{div}((b - \tilde{A})\varphi)(x) dx ds \\
& = \int_t^T \int_{\mathbb{R}^d} f(x, u(s, x), (\sigma^* \nabla u)(s, x)) \varphi(x) dx ds \\
& - \sum_{j=1}^{\infty} \int_t^T \int_{\mathbb{R}^d} g_j(x, u(s, x), (\sigma^* \nabla u)(s, x)) \varphi(x) dx d\hat{\beta}_j(s) \quad P - \text{a.s.}
\end{aligned} \tag{1.4}$$

Here  $\tilde{A}_j \triangleq \frac{1}{2} \sum_{i=1}^d \frac{\partial a_{ij}(x)}{\partial x_i}$ , and  $\tilde{A} = (\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_d)^*$ .

**Remark 1.4** *The weak solution of the forward SPDE (1.1) can be defined similarly. Sometimes in this paper, we denote it by  $v(t, v_0)(\cdot)$  to emphasize its dependence on its initial value  $v_0$ .*

For  $k \geq 0$ , we denote by  $C_{l,b}^k$  the set of  $C^k$ -functions whose partial derivatives of order less than or equal to  $k$  are bounded and by  $H_\rho^k$  the  $\rho$ -weighted Sobolev space (See e.g. [3]). In order to connect BDSDEs with SPDEs, the form of BDSDEs should be a kind of FBDSDEs (forward and backward doubly SDEs). So we first let  $X_s^{t,x}$  be a diffusion process given by the solution of the following forward SDE for  $s \geq t$ ,

$$X_s^{t,x} = x + \int_t^s b(X_u^{t,x}) du + \int_t^s \sigma(X_u^{t,x}) dW_u, \tag{1.5}$$

where  $b \in C_{l,b}^2(\mathbb{R}^d; \mathbb{R}^d)$ ,  $\sigma \in C_{l,b}^3(\mathbb{R}^d; \mathbb{R}^d \times \mathbb{R}^d)$ , and for  $0 \leq s < t$ , we regulate  $X_s^{t,x} = x$ . We now construct the measurable metric dynamical system through defining a measurable and measure-preserving shift. Let  $\hat{\theta}_t : \Omega \rightarrow \Omega$ ,  $t \geq 0$ , be a measurable mapping on  $(\Omega, \mathcal{F}, P)$ , defined by

$$\hat{\theta}_t \circ \hat{B}_s = \hat{B}_{s+t} - \hat{B}_t, \quad \hat{\theta}_t \circ W_s = W_{s+t} - W_t.$$

Then for any  $s, t \geq 0$ ,

- (i)  $P \cdot \hat{\theta}_t^{-1} = P$ ;
- (ii)  $\hat{\theta}_0 = I$ , where  $I$  is the identity transformation on  $\Omega$ ;
- (iii)  $\hat{\theta}_s \circ \hat{\theta}_t = \hat{\theta}_{s+t}$ .

For any  $r \geq 0$ ,  $s \geq t$ ,  $x \in \mathbb{R}^d$ , apply  $\hat{\theta}_r$  to SDE (1.5), then we have

$$\hat{\theta}_r \circ X_s^{t,x} = X_{s+r}^{t+r,x} \quad \text{for all } r, s, t, x \quad \text{a.s.} \tag{1.6}$$

The following lemma in [25] is an extension of the equivalence of norm principle given in [15], [4], [3] to the cases when  $\varphi$  and  $\Psi$  are random.

**Lemma 1.5** *(generalized equivalence of norm principle [25]) Let  $\rho$  be a weight function and  $X$  be the diffusion process given above. If  $s \in [t, T]$ ,  $\varphi : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^1$  is independent of  $\mathcal{F}_{t,s}^W$  and  $\varphi \rho^{-1} \in L^1(\Omega \otimes \mathbb{R}^d)$ , then there exist two constants  $c > 0$  and  $C > 0$  such that*

$$cE\left[\int_{\mathbb{R}^d} |\varphi(x)| \rho^{-1}(x) dx\right] \leq E\left[\int_{\mathbb{R}^d} |\varphi(X_s^{t,x})| \rho^{-1}(x) dx\right] \leq CE\left[\int_{\mathbb{R}^d} |\varphi(x)| \rho^{-1}(x) dx\right].$$

Moreover if  $\Psi : \Omega \times [t, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^1$ ,  $\Psi(s, \cdot)$  is independent of  $\mathcal{F}_{t,s}^W$  and  $\Psi \rho^{-1} \in L^1(\Omega \otimes [t, T] \otimes \mathbb{R}^d)$ , then

$$\begin{aligned}
 cE\left[\int_t^T \int_{\mathbb{R}^d} |\Psi(s, x)|\rho^{-1}(x)dxds\right] &\leq E\left[\int_t^T \int_{\mathbb{R}^d} |\Psi(s, X_s^{t,x})|\rho^{-1}(x)dxds\right] \\
 &\leq CE\left[\int_t^T \int_{\mathbb{R}^d} |\Psi(s, x)|\rho^{-1}(x)dxds\right].
 \end{aligned}$$

We consider the following infinite horizon BDSDE:

$$\begin{aligned}
 e^{-Ks}Y_s^{t,x} &= \int_s^\infty e^{-Kr}f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr + \int_s^\infty Ke^{-Kr}Y_r^{t,x}dr \\
 &\quad - \int_s^\infty e^{-Kr}g(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})d^\dagger \hat{B}_r - \int_s^\infty e^{-Kr}\langle Z_r^{t,x}, dW_r \rangle. \tag{1.7}
 \end{aligned}$$

Here  $\hat{B}_r = \sum_{j=1}^\infty \sqrt{\lambda_j} \hat{\beta}_j(r) e_j$ ,  $\{\hat{\beta}_j(r)\}_{j=1,2,\dots}$  are mutually independent one-dimensional Brownian motions;  $f : \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \rightarrow \mathbb{R}^1$ ;  $g : \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \rightarrow \mathcal{L}_{U_0}^2(\mathbb{R}^1)$ . Set  $g_j \triangleq g\sqrt{\lambda_j}e_j : \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \rightarrow \mathbb{R}^1$ , then Eq.(1.7) is equivalent to

$$\begin{aligned}
 e^{-Ks}Y_s^{t,x} &= \int_s^\infty e^{-Kr}f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr + \int_s^\infty Ke^{-Kr}Y_r^{t,x}dr \\
 &\quad - \sum_{j=1}^\infty \int_s^\infty e^{-Kr}g_j(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})d^\dagger \hat{\beta}_j(r) - \int_s^\infty e^{-Kr}\langle Z_r^{t,x}, dW_r \rangle.
 \end{aligned}$$

**Definition 1.6** (Definition 2.7 in [25]) *A pair of processes  $(Y^{t,\cdot}, Z^{t,\cdot}) \in S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$  is called a solution of Eq.(1.7) if for an arbitrary  $\varphi \in C_c^0(\mathbb{R}^d; \mathbb{R}^1)$ ,*

$$\begin{aligned}
 \int_{\mathbb{R}^d} e^{-Ks}Y_s^{t,x}\varphi(x)dx &= \int_s^\infty \int_{\mathbb{R}^d} e^{-Kr}f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\varphi(x)dxdr \\
 &\quad + \int_s^\infty \int_{\mathbb{R}^d} Ke^{-Kr}Y_r^{t,x}\varphi(x)dxdr \\
 &\quad - \sum_{j=1}^\infty \int_s^\infty \int_{\mathbb{R}^d} e^{-Kr}g_j(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\varphi(x)dx d^\dagger \hat{\beta}_j(r) \\
 &\quad - \int_s^\infty \langle \int_{\mathbb{R}^d} e^{-Kr}Z_r^{t,x}\varphi(x)dx, dW_r \rangle \quad P - \text{a.s.} \tag{1.8}
 \end{aligned}$$

We will prove the following theorem under a general setting.

**Theorem 1.7** *If Eq.(1.7) has a unique solution  $(Y^{t,\cdot}, Z^{t,\cdot}) \in S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$  and  $u(t, \cdot) \triangleq Y_t^{t,\cdot}$  is a continuous weak solution of Eq.(1.3), then  $u(t, \cdot)$  has an indistinguishable version which is a “perfect” stationary weak solution of Eq.(1.3). Furthermore, let  $\hat{B}_s = B_{T'-s} - B_{T'}$  for all  $s \geq 0$  in Eq. (1.7) and  $v_t(\cdot) \triangleq u(T' - t, \cdot) = Y_{T'-t}^{T'-t,\cdot}$  for arbitrary  $T'$  and  $t \in [0, T']$ , then  $v_t(\cdot)$  is independent of  $T'$  and is a “perfect” stationary weak solution of Eq.(1.1) i.e.*

$$v_t(\omega) = v_0(\theta_t\omega) = v(t, v_0(\omega), \omega) \text{ for all } t \geq 0 \text{ a.s.}$$

We will give a proof of this theorem in the last section. In order to find a stationary weak solution of SPDE (1.1), we need to assume reasonable conditions on  $f$  and  $g$  so that we can check the conditions in Theorem 1.7. Indeed under the weak Lipschitz and monotone conditions posed in [25], all the conditions of this theorem can be verified. In this paper, we will consider the following conditions:

(A.1). Functions  $f$  and  $g$  are  $\mathcal{B}_{\mathbb{R}^d} \otimes \mathcal{B}_{\mathbb{R}^1} \otimes \mathcal{B}_{\mathbb{R}^d}$  measurable and there exist constants  $M, M_j, C, C_j, \alpha_j \geq 0$  with  $\sum_{j=1}^{\infty} M_j < \infty$ ,  $\sum_{j=1}^{\infty} C_j < \infty$  and  $\sum_{j=1}^{\infty} \alpha_j < \frac{1}{2}$  s.t. for any  $Y \in L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^1)$ ,  $X_1, X_2, Z_1, Z_2 \in L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^d)$ , and a measurable function  $U : \mathbb{R}^d \rightarrow [0, 1]$ ,

$$\begin{aligned} & \int_{\mathbb{R}^d} U(x) |f(X_1(x), Y(x), Z_1(x)) - f(X_2(x), Y(x), Z_2(x))|^2 \rho^{-1}(x) dx \\ & \leq \int_{\mathbb{R}^d} U(x) (M |X_1(x) - X_2(x)|^2 + C |Z_1(x) - Z_2(x)|^2) \rho^{-1}(x) dx, \\ & \int_{\mathbb{R}^d} U(x) |g_j(X_1(x), Y_1(x), Z_1(x)) - g_j(X_2(x), Y_2(x), Z_2(x))|^2 \rho^{-1}(x) dx \\ & \leq \int_{\mathbb{R}^d} U(x) (M_j |X_1(x) - X_2(x)|^2 + C_j |Y_1(x) - Y_2(x)|^2 + \alpha_j |Z_1(x) - Z_2(x)|^2) \rho^{-1}(x) dx. \end{aligned}$$

(A.2). For  $p \in (2, q - 1)$ ,  $\int_{\mathbb{R}^d} \|g(x, 0, 0)\|_{\mathcal{L}^p_{V_0}(\mathbb{R}^1)}^p \rho^{-1}(x) dx < \infty$ .

(A.3). There exists a constant  $M_0 \geq 0$  s.t. for any  $t \geq 0$ ,  $x, z \in \mathbb{R}^d$ ,  $y \in \mathbb{R}^1$ ,

$$|f(x, y, z)| \leq M_0(1 + |y| + |z|).$$

(A.4). There exists a constant  $\mu > 0$  with  $2\mu - pK - pC - \frac{p(p-1)}{2} \sum_{j=1}^{\infty} C_j > 0$  s.t. for any  $Y_1, Y_2 \in L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^1)$ ,  $X, Z \in L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^d)$ , and a measurable function  $U : \mathbb{R}^d \rightarrow [0, 1]$ ,

$$\begin{aligned} & \int_{\mathbb{R}^d} U(x) (Y_1(x) - Y_2(x)) (f(X(x), Y_1(x), Z(x)) - f(X(x), Y_2(x), Z(x))) \rho^{-1}(x) dx \\ & \leq -\mu \int_{\mathbb{R}^d} U(x) |Y_1(x) - Y_2(x)|^2 \rho^{-1}(x) dx. \end{aligned}$$

(A.5). For any  $x \in \mathbb{R}^d$ ,  $(y, z) \rightarrow f(x, y, z)$  is continuous.

(A.6). The functions  $b \in C^2_{l,b}(\mathbb{R}^d; \mathbb{R}^1)$ ,  $\sigma \in C^3_{l,b}(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^1)$ , and for  $p$  given in (A.2), the global Lipschitz constant  $L$  for  $b$  and  $\sigma$  satisfies  $K - pL - \frac{p(p-1)}{2} L^2 > 0$ .

Note here we don't assume  $f$  is Lipschitz in the variable  $y$ . We will prove

**Theorem 1.8** *Under Conditions (A.1)–(A.6), Eq.(1.7) has a unique solution  $(Y_s^{t,x}, Z_s^{t,x})$ . Moreover  $E[\sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} |Y_s^{t,x}|^p \rho^{-1}(x) dx] < \infty$ .*

**Theorem 1.9** *Under Conditions (A.1)–(A.6), let  $u(t, \cdot) \triangleq Y_t^{t, \cdot}$ , where  $(Y^{t, \cdot}, Z^{t, \cdot})$  is the solution of Eq.(1.7). Then for arbitrary  $T$  and  $t \in [0, T]$ ,  $u(t, \cdot)$  is a weak solution for Eq.(1.3). Moreover,  $u(t, \cdot)$  is a.s. continuous w.r.t.  $t$  in  $L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^1)$ .*

It is obvious that the conditions of Theorem 1.7 are satisfied from the conclusions of Theorem 1.9, so we obtain the stationary weak solution of SPDE (1.1):

**Corollary 1.10** *Under Conditions (A.1)–(A.6), for arbitrary  $T$  and  $t \in [0, T]$ , let  $v(t, \cdot) \triangleq Y_{T-t}^{T-t, \cdot}$ , where  $(Y^{t, \cdot}, Z^{t, \cdot})$  is the solution of Eq.(1.7) with  $\hat{B}_s = B_{T-s} - B_T$  for all  $s \geq 0$ . Then  $v(t, \cdot)$  is a “perfect” stationary weak solution of Eq.(1.1).*

In order to study infinite horizon BDSDEs and stationary solutions of SPDEs, first we have to study the finite time BDSDEs and therefore obtain a probabilistic representation of weak solutions of corresponding SPDEs. This will be given in Section 2. These results are novel, not only because of the connection of BDSDEs and SPDEs, but also due to the fact that the SPDEs we study appear to be new as coefficient  $g$  of the noise can be a general one. The existence and

uniqueness of such equations when  $g$  is independent of  $\nabla v$  or linearly dependent on  $\nabla v$  were studied by the pioneering works of Da Prato and Zabczyk [8], Krylov [13]. Our work shows that studying the BDSDEs is a natural method for studying such general SPDEs. The infinite horizon BDSDEs and stationary solution of SPDEs will be studied in Section 3.

## 2 Finite Horizon BDSDEs and the Corresponding SPDEs

### 2.1 Conditions and main results

In this section, we will study the following BDSDEs on finite horizon and establish its connection with SPDEs, which is necessary to establish the solution of infinite horizon BDSDE and its connection with the SPDEs:

$$\begin{aligned} Y_s^{t,x} &= h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr \\ &\quad - \int_s^T g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) d^\dagger \hat{B}_r - \int_s^T \langle Z_r^{t,x}, dW_r \rangle, \quad 0 \leq s \leq T. \end{aligned} \quad (2.1)$$

Here  $h : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^1$ ,  $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \rightarrow \mathbb{R}^1$ ,  $g : [0, T] \times \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \rightarrow \mathcal{L}_{U_0}^2(\mathbb{R}^1)$ . Set  $g_j \triangleq g\sqrt{\lambda_j}e_j : [0, T] \times \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \rightarrow \mathbb{R}^1$ , then Eq.(2.1) is equivalent to

$$\begin{aligned} Y_s^{t,x} &= h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr \\ &\quad - \sum_{j=1}^{\infty} \int_s^T g_j(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) d^\dagger \hat{\beta}_j(r) - \int_s^T \langle Z_r^{t,x}, dW_r \rangle, \quad 0 \leq s \leq T. \end{aligned}$$

**Definition 2.1** A pair of processes  $(Y^{t,\cdot}, Z^{t,\cdot}) \in S^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$  is called a solution of Eq.(2.1) if for any  $\varphi \in C_c^0(\mathbb{R}^d; \mathbb{R}^1)$ ,

$$\begin{aligned} \int_{\mathbb{R}^d} Y_s^{t,x} \varphi(x) dx &= \int_{\mathbb{R}^d} h(X_T^{t,x}) \varphi(x) dx + \int_s^T \int_{\mathbb{R}^d} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \varphi(x) dx dr \\ &\quad - \sum_{j=1}^{\infty} \int_s^T \int_{\mathbb{R}^d} g_j(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \varphi(x) dx d^\dagger \hat{\beta}_j(r) \\ &\quad - \int_s^T \left\langle \int_{\mathbb{R}^d} Z_r^{t,x} \varphi(x) dx, dW_r \right\rangle \quad P - \text{a.s.} \end{aligned} \quad (2.2)$$

We assume

- (H.1). Function  $h$  is  $\mathcal{F}_{T,\infty}^{\hat{B}} \otimes \mathcal{B}_{\mathbb{R}^d}$  measurable and  $E[\int_{\mathbb{R}^d} |h(x)|^2 \rho^{-1}(x) dx] < \infty$ .  
 (H.2). Functions  $f$  and  $g$  are  $\mathcal{B}_{[0,T]} \otimes \mathcal{B}_{\mathbb{R}^d} \otimes \mathcal{B}_{\mathbb{R}^1} \otimes \mathcal{B}_{\mathbb{R}^d}$  measurable and there exist constants  $C, C_j, \alpha_j \geq 0$  with  $\sum_{j=1}^{\infty} C_j < \infty$  and  $\sum_{j=1}^{\infty} \alpha_j < \frac{1}{2}$  s.t. for any  $r \in [0, T]$ ,  $Y, Y_1, Y_2 \in L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)$ ,  $X, Z_1, Z_2 \in L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)$ ,

$$\begin{aligned} &\int_{\mathbb{R}^d} |f(r, X(x), Y(x), Z_1(x)) - f(r, X(x), Y(x), Z_2(x))|^2 \rho^{-1}(x) dx \\ &\leq C \int_{\mathbb{R}^d} |Z_1(x) - Z_2(x)|^2 \rho^{-1}(x) dx, \end{aligned}$$

$$\begin{aligned} & \int_{\mathbb{R}^d} |g_j(r, X(x), Y_1(x), Z_1(x)) - g_j(r, X(x), Y_2(x), Z_2(x))|^2 \rho^{-1}(x) dx \\ & \leq \int_{\mathbb{R}^d} (C_j |Y_1(x) - Y_2(x)|^2 + \alpha_j |Z_1(x) - Z_2(x)|^2) \rho^{-1}(x) dx. \end{aligned}$$

(H.3). The integral  $\int_0^T \int_{\mathbb{R}^d} \|g(r, x, 0, 0)\|_{\mathcal{L}_{V_0}^2(\mathbb{R}^1)}^2 \rho^{-1}(x) dx dr < \infty$ .

(H.4). There exists a constant  $M'_0 \geq 0$  s.t. for any  $r \in [0, T]$ ,  $x, z \in \mathbb{R}^d$ ,  $y \in \mathbb{R}^1$ ,

$$|f(r, x, y, z)| \leq M'_0(1 + |y| + |z|).$$

(H.5). There exists a constant  $\mu \in \mathbb{R}^1$  s.t. for any  $r \in [0, T]$ ,  $Y_1, Y_2 \in L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)$ ,  $X, Z \in L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)$ , and a measurable function  $U : \mathbb{R}^d \rightarrow [0, 1]$ ,

$$\begin{aligned} & \int_{\mathbb{R}^d} U(x) (Y_1(x) - Y_2(x)) (f(r, X(x), Y_1(x), Z(x)) - f(r, X(x), Y_2(x), Z(x))) \rho^{-1}(x) dx \\ & \leq \mu \int_{\mathbb{R}^d} U(x) |Y_1(x) - Y_2(x)|^2 \rho^{-1}(x) dx. \end{aligned}$$

(H.6). For any  $r \in [0, T]$ ,  $x \in \mathbb{R}^d$ ,  $(y, z) \rightarrow f(r, x, y, z)$  is continuous.

(H.7). The functions  $b \in C_{l,b}^2(\mathbb{R}^d; \mathbb{R}^d)$ ,  $\sigma \in C_{l,b}^3(\mathbb{R}^d; \mathbb{R}^d \times \mathbb{R}^d)$ .

The first objective of this section is to prove

**Theorem 2.2** *Under Conditions (H.1)–(H.7), BDSDE (2.1) has a unique solution.*

Then we will make the connection between the solutions of BDSDE (2.1) and SPDE (1.3).

**Theorem 2.3** *Under Conditions (H.1)–(H.7), if we define  $u(t, x) = Y_t^{t,x}$ , where  $(Y_s^{t,x}, Z_s^{t,x})$  is the solution of Eq.(2.1), then  $u(t, x)$  is the unique weak solution of Eq.(1.3) with  $u(T, x) = h(x)$ . Moreover,  $u(s, X_s^{t,x}) = Y_s^{t,x}$ ,  $(\sigma^* \nabla u)(s, X_s^{t,x}) = Z_s^{t,x}$  for a.a.  $s \in [t, T]$ ,  $x \in \mathbb{R}^d$  a.s.*

## 2.2 Existence and uniqueness of the solutions of BDSDEs with finite dimensional noise

In their pioneering work [19], Pardoux and Peng solved the following BSDE with Lipschitz conditions on the coefficient:

$$Y_s = \xi + \int_s^T f(r, Y_r, Z_r) dr - \int_s^T \langle Z_r, dW_r \rangle. \quad (2.3)$$

After that, many researchers studied how to weaken the Lipschitz conditions so that the BSDE system can include more equations. To name but a few, in [21], [16], [18], [12], [5] and [6], researchers made their significant contributions to this subject. In [16], Lepeltier and San Martin assumed that the  $\mathbb{R}^1$ -valued function  $f(r, y, z)$  satisfies the measurable condition, the  $y, z$  linear growth condition and the  $y, z$  continuous condition, then they proved the existence of the solution of Eq.(2.3). But the uniqueness of solution failed to be proved since the comparison theorem cannot be used under non-Lipschitz condition.

In [22], after proving the comparison theorem of BDSDE with Lipschitz condition, the authors used the method in [16] and proved the corresponding result for the following  $\mathbb{R}^1$ -valued BDSDE:

$$Y_s = \xi + \int_s^T f(r, Y_r, Z_r) dr - \int_s^T \langle g(r, Y_r, Z_r), d^\dagger \hat{B}_r \rangle - \int_s^T \langle Z_r, dW_r \rangle. \quad (2.4)$$

They assumed the same condition for  $f$  as in [16] and  $g(r, y, z)$  satisfies the standard measurable condition and Lipschitz condition w.r.t.  $y$  and  $z$ . Then in Theorem 4.1 in [22], they proved the existence of solution of Eq.(2.4).

First we study the following BDSDE with finite dimensional noise under non-Lipschitz conditions:

$$\begin{aligned} Y_s^{t,x,n} &= h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) dr \\ &\quad - \sum_{j=1}^n \int_s^T g_j(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) d^\dagger \hat{\beta}_j(r) - \int_s^T \langle Z_r^{t,x,n}, dW_r \rangle. \end{aligned} \quad (2.5)$$

Note here in [22] and [16], the authors only dealt with the solution of Eq.(2.5) for a fixed  $x$  almost surely. Of course if one is interested in the classical solution of this SPDEs, it is easy to see that this implies one can solve Eq.(2.5) for all  $x \in \mathbb{R}^d$  a.s. by some standard perfection arguments. But we consider the solution in the space  $S^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$  in order to consider the weak solution of the SPDEs. The main task of this subsection is to prove

**Theorem 2.4** *Under Conditions (H.1)–(H.7), Eq.(2.5) has a unique solution*

$$(Y_s^{t,\cdot,n}, Z_s^{t,\cdot,n}) \in S^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)).$$

We will first acknowledge the following Proposition 2.5 at the moment, then we prove Theorem 2.4 with the help of Proposition 2.5. Note that in the proof of Theorem 2.4 and Proposition 2.5, we can consider the solution in  $S^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$  due to the arguments in Remark 3.7 in [25].

**Proposition 2.5** *Given  $(U(\cdot), V(\cdot)) \in S^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$ , then under Conditions (H.1)–(H.7), the equation*

$$\begin{aligned} Y_s^{t,x,n} &= h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) dr \\ &\quad - \sum_{j=1}^n \int_s^T g_j(r, X_r^{t,x}, U_r(x), V_r(x)) d^\dagger \hat{\beta}_j(r) - \int_s^T \langle Z_r^{t,x,n}, dW_r \rangle \end{aligned} \quad (2.6)$$

*has a unique solution.*

*Proof of Theorem 2.4. Uniqueness.* Assume there exists another  $(\hat{Y}_s^{t,\cdot,n}, \hat{Z}_s^{t,\cdot,n}) \in S^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$  satisfying (2.5). Define

$$\bar{Y}_s^{t,x,n} = Y_s^{t,x,n} - \hat{Y}_s^{t,x,n} \quad \text{and} \quad \bar{Z}_s^{t,x,n} = Z_s^{t,x,n} - \hat{Z}_s^{t,x,n}, \quad t \leq s \leq T.$$

Then with probability 1 we have that for a.e.  $x \in \mathbb{R}^d$ ,  $(\bar{Y}_s^{t,x,n}, \bar{Z}_s^{t,x,n})$  satisfies

$$\begin{aligned} \bar{Y}_s^{t,x,n} &= \int_s^T (f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) - f(r, X_r^{t,x}, \hat{Y}_s^{t,x,n}, \hat{Z}_s^{t,x,n})) dr \\ &\quad - \sum_{j=1}^n \int_s^T (g_j(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) - g_j(r, X_r^{t,x}, \hat{Y}_s^{t,x,n}, \hat{Z}_s^{t,x,n})) d^\dagger \hat{\beta}_j(r) \\ &\quad - \int_s^T \langle \bar{Z}_r^{t,x,n}, dW_r \rangle. \end{aligned}$$

From Condition (H.4) and  $(\hat{Y}^{t,\cdot,n}, \hat{Z}^{t,\cdot,n}), (Y^{t,\cdot,n}, Z^{t,\cdot,n}) \in S^{2,0}([t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^d))$ , it follows that

$$\begin{aligned} & E\left[\int_t^T \int_{\mathbb{R}^d} |f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) - f(r, X_r^{t,x}, \hat{Y}_s^{t,x,n}, \hat{Z}_s^{t,x,n})|^2 \rho^{-1}(x) dx dr\right] \\ & \leq C_p E\left[\int_t^T \int_{\mathbb{R}^d} (1 + |Y_r^{t,x,n}|^2 + |\hat{Y}_r^{t,x,n}|^2 + |Z_r^{t,x,n}|^2 + |\hat{Z}_r^{t,x,n}|^2) \rho^{-1}(x) dx dr\right] < \infty, \end{aligned}$$

where and in the rest of this paper  $C_p$  is a generic constant. So from Fubini theorem we have for a.e.  $x \in \mathbb{R}^d$ ,

$$E\left[\int_t^T |f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) - f(r, X_r^{t,x}, \hat{Y}_r^{t,x,n}, \hat{Z}_r^{t,x,n})|^2 dr\right] < \infty.$$

Similarly, with Condition (H.2), we have for a.e.  $x \in \mathbb{R}^d$ ,

$$\sum_{j=1}^n E\left[\int_t^T |g_j(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) - g_j(r, X_r^{t,x}, \hat{Y}_r^{t,x,n}, \hat{Z}_r^{t,x,n})|^2 dr\right] < \infty.$$

For a.e.  $x \in \mathbb{R}^d$ , we apply the generalized Itô's formula ([10]) to  $e^{Ks}\psi_M(\bar{Y}_s^{t,x,n})$ , where  $K \in \mathbb{R}^1$  and

$$\psi_M(x) = x^2 I_{\{-M \leq x < M\}} + M(2x - M) I_{\{x \geq M\}} - M(2x + M) I_{\{x < -M\}}.$$

Then

$$\begin{aligned} & e^{Ks}\psi_M(\bar{Y}_s^{t,x,n}) + K \int_s^T e^{Kr} \psi_M(\bar{Y}_r^{t,x,n}) dr + \int_s^T e^{Kr} I_{\{-M \leq \bar{Y}_r^{t,x,n} < M\}} |\bar{Z}_r^{t,x,n}|^2 dr \\ & = \int_s^T e^{Kr} \psi'_M(\bar{Y}_r^{t,x,n}) (f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) - f(r, X_r^{t,x}, \hat{Y}_r^{t,x,n}, \hat{Z}_r^{t,x,n})) dr \\ & \quad + \sum_{j=1}^n \int_s^T e^{Kr} I_{\{-M \leq \bar{Y}_r^{t,x,n} < M\}} |g_j(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) - g_j(r, X_r^{t,x}, \hat{Y}_r^{t,x,n}, \hat{Z}_r^{t,x,n})|^2 dr \\ & \quad - \sum_{j=1}^n \int_s^T e^{Kr} \psi'_M(\bar{Y}_r^{t,x,n}) (g_j(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) - g_j(r, X_r^{t,x}, \hat{Y}_r^{t,x,n}, \hat{Z}_r^{t,x,n})) d^\dagger \hat{\beta}_j(r) \\ & \quad - \int_s^T \langle e^{Kr} \psi'_M(\bar{Y}_r^{t,x,n}) \bar{Z}_r^{t,x,n}, dW_r \rangle. \end{aligned} \tag{2.7}$$

We can use the Fubini theorem to perfect (2.7) so that (2.7) is satisfied for a.e.  $x \in \mathbb{R}^d$  on a full measure set that is independent of  $x$ . If we define  $\frac{\psi'_M(x)}{x} = 2$  when  $x = 0$ , then  $0 \leq \frac{\psi'_M(\bar{Y}_r^{t,x,n})}{\bar{Y}_r^{t,x,n}} \leq 2$ . Taking integration over  $\mathbb{R}^d$  on both sides of (2.7), we can apply the stochastic Fubini theorem ([8]). Noting that the stochastic integrals are martingales, so taking the expectation, we have

$$\begin{aligned} & E\left[\int_{\mathbb{R}^d} e^{Ks}\psi_M(\bar{Y}_s^{t,x,n}) \rho^{-1}(x) dx\right] + KE\left[\int_s^T \int_{\mathbb{R}^d} e^{Kr} \psi_M(\bar{Y}_r^{t,x,n}) \rho^{-1}(x) dx dr\right] \\ & \quad + E\left[\int_s^T \int_{\mathbb{R}^d} e^{Kr} I_{\{-M \leq \bar{Y}_r^{t,x,n} < M\}} |\bar{Z}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right] \\ & \leq E\left[\int_s^T \int_{\mathbb{R}^d} e^{Kr} \frac{\psi'_M(\bar{Y}_r^{t,x,n})}{\bar{Y}_r^{t,x,n}} \mu |\bar{Y}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right] \end{aligned}$$

$$\begin{aligned}
 & + (2C + \sum_{j=1}^{\infty} C_j) E \left[ \int_s^T \int_{\mathbb{R}^d} e^{Kr} |\bar{Y}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr \right] \\
 & + \left( \frac{1}{2} + \sum_{j=1}^{\infty} \alpha_j \right) E \left[ \int_s^T \int_{\mathbb{R}^d} e^{Kr} |\bar{Z}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr \right].
 \end{aligned}$$

Taking the limit as  $M \rightarrow \infty$  and applying the monotone convergence theorem, we have

$$\begin{aligned}
 & E \left[ \int_{\mathbb{R}^d} e^{Ks} |\bar{Y}_s^{t,x,n}|^2 \rho^{-1}(x) dx \right] + \left( \frac{1}{2} - \sum_{j=1}^{\infty} \alpha_j \right) E \left[ \int_s^T \int_{\mathbb{R}^d} e^{Kr} |\bar{Z}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr \right] \\
 & + (K - 2\mu - 2C - \sum_{j=1}^{\infty} C_j) E \left[ \int_s^T \int_{\mathbb{R}^d} e^{Kr} |\bar{Y}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr \right] \leq 0. \tag{2.8}
 \end{aligned}$$

Note that all the terms on the left hand side of (2.8) are positive when  $K$  is taken sufficiently large. So by a ‘‘standard’’ argument, we have  $\bar{Y}_s^{t,x,n} = 0$  for all  $s \in [t, T]$ , a.a.  $x \in \mathbb{R}^d$  a.s. Also by (2.8), for a.e.  $\bar{Z}_s^{t,x,n} = 0$  for a.a.  $x \in \mathbb{R}^d$  a.s. We can modify the values of  $Z$  at the measure zero exceptional set of  $s$  such that  $\bar{Z}_s^{t,x,n} = 0$  for all  $x \in \mathbb{R}^d$  a.s.

**Existence.** If we regard Eq.(2.6) as a mapping, then by Proposition 2.5, (2.6) is an iterated mapping from  $S^{2,0}([t, T]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([t, T]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^d))$  to itself and we can obtain a sequence  $\{(Y_r^{t,x,n,i}, Z_r^{t,x,n,i})\}_{i=1}^{\infty}$  from this mapping. We will prove that (2.6) is a contraction mapping. For this, define for  $t \leq s \leq T$  and arbitrary given  $(Y^{\cdot, \cdot, n, 1}, Z^{\cdot, \cdot, n, 1}) \in S^{2,0}([t, T]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([t, T]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^d))$ ,

$$\begin{aligned}
 \bar{Y}_s^{t,x,n,i} &= Y_s^{t,x,n,i} - Y_s^{t,x,n,i-1}, \quad \bar{Z}_s^{t,x,n,i} = Z_s^{t,x,n,i} - Z_s^{t,x,n,i-1}, \\
 \bar{g}_j^i(s, x) &= g_j(s, X_s^{t,x}, Y_s^{t,x,n,i}, Z_s^{t,x,n,i}) - g_j(s, X_s^{t,x}, Y_s^{t,x,n,i-1}, Z_s^{t,x,n,i-1}), \quad i = 2, 3, \dots
 \end{aligned}$$

Then for a.e.  $x \in \mathbb{R}^d$ ,  $(\bar{Y}_s^{t,x,n,N}, \bar{Z}_s^{t,x,n,N})$  satisfies

$$\begin{aligned}
 \bar{Y}_s^{t,x,n,N} &= \int_s^T (f(r, X_r^{t,x}, Y_r^{t,x,n,N}, Z_r^{t,x,n,N}) - f(r, X_r^{t,x}, Y_s^{t,x,n,N-1}, Z_s^{t,x,n,N-1})) dr \\
 &\quad - \sum_{j=1}^n \int_s^T \bar{g}_j^{N-1}(r, x) d\hat{\beta}_j(r) - \int_s^T \langle \bar{Z}_r^{t,x,n,N}, dW_r \rangle.
 \end{aligned}$$

Applying generalized Itô’s formula to  $e^{Kr} \psi_M(\bar{Y}_r^{t,x,n,N})$  for a.e.  $x \in \mathbb{R}^d$ , by the Young inequality, Conditions (H.2) and (H.5), we can deduce that

$$\begin{aligned}
 & \int_{\mathbb{R}^d} e^{Ks} \psi_M(\bar{Y}_s^{t,x,n,N}) \rho^{-1}(x) dx + K \int_s^T \int_{\mathbb{R}^d} e^{Kr} \psi_M(\bar{Y}_r^{t,x,n,N}) \rho^{-1}(x) dx dr \\
 & + \int_s^T \int_{\mathbb{R}^d} e^{Kr} I_{\{-M \leq \bar{Y}_r^{t,x,n,N} < M\}} |\bar{Z}_r^{t,x,n,N}|^2 \rho^{-1}(x) dx dr \\
 & \leq \int_s^T \int_{\mathbb{R}^d} e^{Kr} \frac{\psi'_M(\bar{Y}_r^{t,x,n,N})}{\bar{Y}_r^{t,x,n,N}} \mu |\bar{Y}_r^{t,x,n,N}|^2 \rho^{-1}(x) dx dr \\
 & + 2C \int_s^T \int_{\mathbb{R}^d} e^{Kr} |\bar{Y}_r^{t,x,n,N}|^2 \rho^{-1}(x) dx dr + \frac{1}{2} \int_s^T \int_{\mathbb{R}^d} e^{Kr} |\bar{Z}_r^{t,x,n,N}|^2 \rho^{-1}(x) dx dr \\
 & + \sum_{j=1}^{\infty} C_j \int_s^T \int_{\mathbb{R}^d} e^{Kr} |\bar{Y}_r^{t,x,n,N-1}|^2 \rho^{-1}(x) dx dr + \sum_{j=1}^{\infty} \alpha_j \int_s^T \int_{\mathbb{R}^d} e^{Kr} |\bar{Z}_r^{t,x,n,N-1}|^2 \rho^{-1}(x) dx dr
 \end{aligned}$$

$$\begin{aligned}
& - \sum_{j=1}^n \int_s^T \int_{\mathbb{R}^d} e^{Kr} \psi'_M(\bar{Y}_r^{t,x,n,N}) \bar{g}_j^{N-1}(r,x) \rho^{-1}(x) dx d\hat{\beta}_j(r) \\
& - \int_s^T \left\langle \int_{\mathbb{R}^d} e^{Kr} \psi'_M(\bar{Y}_r^{t,x,n,N}) \bar{Z}_r^{t,x,n,N} \rho^{-1}(x) dx, dW_r \right\rangle.
\end{aligned} \tag{2.9}$$

Then taking expectation and the limit as  $M \rightarrow \infty$ , we have

$$\begin{aligned}
& (K - 2\mu - 2C)E\left[\int_s^T \int_{\mathbb{R}^d} e^{Kr} |\bar{Y}_r^{t,x,n,N}|^2 \rho^{-1}(x) dx dr\right] + \frac{1}{2}E\left[\int_s^T \int_{\mathbb{R}^d} e^{Kr} |\bar{Z}_r^{t,x,n,N}|^2 \rho^{-1}(x) dx dr\right] \\
& \leq E\left[\int_s^T \int_{\mathbb{R}^d} e^{Kr} \left(\sum_{j=1}^{\infty} C_j |\bar{Y}_r^{t,x,n,N-1}|^2 + \sum_{j=1}^{\infty} \alpha_j |\bar{Z}_r^{t,x,n,N-1}|^2\right) \rho^{-1}(x) dx dr\right].
\end{aligned}$$

First assuming that  $\sum_{j=1}^{\infty} C_j, \sum_{j=1}^{\infty} \alpha_j > 0$ , we have

$$\begin{aligned}
& (2K - 4\mu - 4C)E\left[\int_s^T \int_{\mathbb{R}^d} e^{Kr} |\bar{Y}_r^{t,x,n,N}|^2 \rho^{-1}(x) dx dr\right] + E\left[\int_s^T \int_{\mathbb{R}^d} e^{Kr} |\bar{Z}_r^{t,x,n,N}|^2 \rho^{-1}(x) dx dr\right] \\
& \leq 2 \sum_{j=1}^{\infty} \alpha_j E\left[\int_s^T \int_{\mathbb{R}^d} e^{Kr} \left(\frac{\sum_{j=1}^{\infty} C_j}{\sum_{j=1}^{\infty} \alpha_j} |\bar{Y}_r^{t,x,n,N-1}|^2 + |\bar{Z}_r^{t,x,n,N-1}|^2\right) \rho^{-1}(x) dx dr\right].
\end{aligned}$$

Letting  $K = 2\mu + 2C + \frac{\sum_{j=1}^{\infty} C_j}{2 \sum_{j=1}^{\infty} \alpha_j}$ , we have

$$\begin{aligned}
& E\left[\int_s^T \int_{\mathbb{R}^d} e^{Kr} \left(\frac{\sum_{j=1}^{\infty} C_j}{\sum_{j=1}^{\infty} \alpha_j} |\bar{Y}_r^{t,x,n,N}|^2 + |\bar{Z}_r^{t,x,n,N}|^2\right) \rho^{-1}(x) dx dr\right] \\
& \leq 2 \sum_{j=1}^{\infty} \alpha_j E\left[\int_s^T \int_{\mathbb{R}^d} e^{Kr} \left(\frac{\sum_{j=1}^{\infty} C_j}{\sum_{j=1}^{\infty} \alpha_j} |\bar{Y}_r^{t,x,n,N-1}|^2 + |\bar{Z}_r^{t,x,n,N-1}|^2\right) \rho^{-1}(x) dx dr\right].
\end{aligned} \tag{2.10}$$

Note that  $E\left[\int_t^T \int_{\mathbb{R}^d} e^{Kr} \left(\frac{\sum_{j=1}^{\infty} C_j}{\sum_{j=1}^{\infty} \alpha_j} |\cdot|^2 + |\cdot|^2\right) \rho^{-1}(x) dx dr\right]$  is equivalent to  $E\left[\int_t^T \int_{\mathbb{R}^d} (|\cdot|^2 + |\cdot|^2) \rho^{-1}(x) dx dr\right]$ . From the contraction principle, the mapping (2.6) has a pair of fixed point  $(Y^{t,\cdot,n,\infty}, Z^{t,\cdot,n,\infty})$  that is the limit of the Cauchy sequence  $\{(Y^{t,\cdot,n,N}, Z^{t,\cdot,n,N})\}_{N=1}^{\infty}$  in  $M^{2,0}([t, T]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([t, T]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^d))$ . We then prove that  $Y^{t,\cdot,n,\infty}$  is also the limit of  $Y^{t,\cdot,n,N}$  in  $S^{2,0}([t, T]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^1))$  as  $N \rightarrow \infty$ . For this, we only need to prove that  $\{Y^{t,\cdot,n,N}\}_{N=1}^{\infty}$  is a Cauchy sequence in  $S^{2,0}([t, T]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^1))$ . For this, from (2.9), by the B-D-G inequality, the Cauchy-Schwartz inequality and the Young inequality, we have

$$\begin{aligned}
& E\left[\sup_{t \leq s \leq T} \int_{\mathbb{R}^d} e^{Ks} \psi_M(\bar{Y}_s^{t,x,n,N}) \rho^{-1}(x) dx\right] \\
& \leq C_p E\left[\int_t^T \int_{\mathbb{R}^d} (|\bar{Y}_r^{t,x,n,N}|^2 + |\bar{Z}_r^{t,x,n,N}|^2 + |\bar{Y}_r^{t,x,n,N-1}|^2 + |\bar{Z}_r^{t,x,n,N-1}|^2) \rho^{-1}(x) dx dr\right] \\
& \quad + C_p E\left[\sqrt{\int_t^T \int_{\mathbb{R}^d} |\psi'_M(\bar{Y}_r^{t,x,n,N})|^2 \rho^{-1}(x) dx} \int_{\mathbb{R}^d} \sum_{j=1}^n |\bar{g}_j^{N-1}(r,x)|^2 \rho^{-1}(x) dx dr\right] \\
& \quad + C_p E\left[\sqrt{\int_t^T \int_{\mathbb{R}^d} |\psi'_M(\bar{Y}_r^{t,x,n,N})|^2 \rho^{-1}(x) dx} \int_{\mathbb{R}^d} |\bar{Z}_r^{t,x,n,N}|^2 \rho^{-1}(x) dx dr\right] \\
& \leq C_p E\left[\int_t^T \int_{\mathbb{R}^d} (|\bar{Y}_r^{t,x,n,N}|^2 + |\bar{Z}_r^{t,x,n,N}|^2 + |\bar{Y}_r^{t,x,n,N-1}|^2 + |\bar{Z}_r^{t,x,n,N-1}|^2) \rho^{-1}(x) dx dr\right]
\end{aligned}$$

$$+ \frac{1}{5} E[ \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\psi'_M(Y_s(x))|^2 \rho^{-1}(x) dx],$$

where  $C_p$  depends on  $|\mu|$ ,  $C$ ,  $\sum_{j=1}^{\infty} \alpha_j$ ,  $\sum_{j=1}^{\infty} C_j$  and the fixed constant in the B-D-G inequality. Taking the limit as  $M \rightarrow \infty$  and applying the monotone convergence theorem, we have

$$\begin{aligned} & E[ \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} e^{Ks} |\bar{Y}_s^{t,x,n,N}|^2 \rho^{-1}(x) dx] \\ & \leq M_0'' E[ \int_s^T \int_{\mathbb{R}^d} e^{Kr} (|\bar{Y}_r^{t,x,n,N-1}|^2 + |\bar{Z}_r^{t,x,n,N-1}|^2 + |\bar{Y}_r^{t,x,n,N}|^2 + |\bar{Z}_r^{t,x,n,N}|^2) \rho^{-1}(x) dx dr], \end{aligned} \quad (2.11)$$

where  $M_0'' > 0$  is independent of  $n$  and  $N$ . Without losing any generality, assume that  $M \geq N$ . We can deduce from (2.10) and (2.11) that

$$\begin{aligned} & (E[ \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |Y_s^{t,x,n,M} - Y_s^{t,x,n,N}|^2 \rho^{-1}(x) dx])^{\frac{1}{2}} \\ & \leq \sum_{i=N+1}^M (E[ \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{Y}_s^{t,x,n,i}|^2 \rho^{-1}(x) dx])^{\frac{1}{2}} \\ & \leq \sum_{i=N+1}^M (M_0'' E[ \int_t^T \int_{\mathbb{R}^d} e^{Kr} (|\bar{Y}_r^{t,x,n,i-1}|^2 + |\bar{Z}_r^{t,x,n,i-1}|^2 + |\bar{Y}_r^{t,x,n,i}|^2 + |\bar{Z}_r^{t,x,n,i}|^2) \rho^{-1}(x) dx dr])^{\frac{1}{2}} \\ & \leq \sum_{i=N+1}^M ((1 + \frac{\sum_{j=1}^{\infty} \alpha_j}{\sum_{j=1}^{\infty} C_j}) M_0'' E[ \int_t^T \int_{\mathbb{R}^d} e^{Kr} (\frac{\sum_{j=1}^{\infty} C_j}{\sum_{j=1}^{\infty} \alpha_j} |\bar{Y}_r^{t,x,n,i-1}|^2 + |\bar{Z}_r^{t,x,n,i-1}|^2 \\ & \quad + \frac{\sum_{j=1}^{\infty} C_j}{\sum_{j=1}^{\infty} \alpha_j} |\bar{Y}_r^{t,x,n,i}|^2 + |\bar{Z}_r^{t,x,n,i}|^2) \rho^{-1}(x) dx dr])^{\frac{1}{2}} \\ & \leq \sum_{i=N+1}^M ((2 + \frac{2 \sum_{j=1}^{\infty} \alpha_j}{\sum_{j=1}^{\infty} C_j}) M_0'' E[ \int_t^T \int_{\mathbb{R}^d} e^{Kr} (\frac{\sum_{j=1}^{\infty} C_j}{\sum_{j=1}^{\infty} \alpha_j} |\bar{Y}_r^{t,x,n,i-1}|^2 + |\bar{Z}_r^{t,x,n,i-1}|^2) \rho^{-1}(x) dx dr])^{\frac{1}{2}} \\ & \leq \sum_{i=N+1}^{\infty} (2 \sum_{j=1}^{\infty} \alpha_j)^{\frac{i-2}{2}} ((2 + \frac{2 \sum_{j=1}^{\infty} \alpha_j}{\sum_{j=1}^{\infty} C_j}) M_0'' \\ & \quad \times E[ \int_t^T \int_{\mathbb{R}^d} e^{Kr} (\frac{\sum_{j=1}^{\infty} C_j}{\sum_{j=1}^{\infty} \alpha_j} |Y_r^{t,x,n,1}|^2 + |Z_r^{t,x,n,1}|^2) \rho^{-1}(x) dx dr])^{\frac{1}{2}} \rightarrow 0 \end{aligned}$$

as  $M, N \rightarrow \infty$ , since  $2 \sum_{j=1}^{\infty} \alpha_j < 1$ . So we proved our claim.

If either or both  $\sum_{j=1}^{\infty} C_j, \sum_{j=1}^{\infty} \alpha_j = 0$ , we can prove the above convergence using similar method or the above convergence is trivially correct. Theorem 2.4 is proved.  $\diamond$

The remaining work in this subsection is to prove Proposition 2.5. First we do some preparations.

**Lemma 2.6** *Under Conditions (H.1)–(H.7), if there exists  $(Y(\cdot), Z(\cdot)) \in M^{2,0}([t, T]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([t, T]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^d))$  satisfying the spatial integral form of Eq.(2.5) for  $t \leq s \leq T$ , then  $Y(\cdot) \in S^{2,0}([t, T]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^1))$  and therefore  $(Y_s(x), Z_s(x))$  is a solution of Eq.(2.5).*

*Proof.* Similar to the proof of Lemma 3.3 in [25], we can prove  $Y_s(\cdot)$  is continuous w.r.t.  $s$  in  $L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^1)$  under the conditions of this lemma. We only mention that we can use Condition (H.4) to deal with the term  $f(r, X_r^{t,x}, Y_r(x), Z_r(x))$  although there is no weak

Lipschitz condition for  $Y_r(x)$ . We omit the proof here. Now we only show the proof of  $E[\sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |Y_s(x)|^2 \rho^{-1}(x) dx] < \infty$  briefly. For a.e.  $x \in \mathbb{R}^d$ , applying the generalized Itô's formula to  $\psi_M(Y_r(x))$ , by Lemma 1.5, the B-D-G inequality and the Cauchy-Schwartz inequality, we have

$$\begin{aligned} & E\left[\sup_{t \leq s \leq T} \int_{\mathbb{R}^d} \psi_M(Y_s(x)) \rho^{-1}(x) dx\right] \\ & \leq C_p E\left[\int_{\mathbb{R}^d} |h(x)|^2 \rho^{-1}(x) dx\right] + C_p E\left[\int_t^T \int_{\mathbb{R}^d} (|Y_r(x)|^2 + |Z_r(x)|^2) \rho^{-1}(x) dx dr\right] \\ & \quad + C_p \sum_{j=1}^n \int_t^T \int_{\mathbb{R}^d} (1 + |g_j(r, x, 0, 0)|^2) \rho^{-1}(x) dx dr < \infty. \end{aligned}$$

So taking the limit as  $M \rightarrow \infty$  and applying the monotone convergence theorem, we have  $Y(\cdot) \in S^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$ . Recall that a solution of Eq.(2.5) is a pair of processes in  $S^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$  satisfying the spatial integral form of Eq.(2.5), therefore  $(Y_s(x), Z_s(x))$  is a solution of Eq.(2.5).  $\diamond$

From the proof of Lemma 2.6, one can similarly deduce that

**Corollary 2.7** *Under Conditions (H.1)–(H.7), if there exists  $(Y(\cdot), Z(\cdot)) \in M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$  satisfying the spatial integral form of Eq.(2.6) for  $t \leq s \leq T$ , then  $Y(\cdot) \in S^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$  and therefore  $(Y_s(x), Z_s(x))$  is a solution of Eq.(2.6).*

For the rest of this paper, we will leave out a similar localization argument as in the proof of Theorem 2.4 and Lemma 2.6 when applying Itô's formula to save the space of this paper.

*Proof of Proposition 2.5.* The proof of the uniqueness is rather similar to the uniqueness proof in Theorem 2.4, so it is omitted.

Existence. Define

$$\tilde{f}^x(r, y, z) = f(r, X_r^{t,x}, y, z) \text{ and } \tilde{g}_j^x(r) = g_j(r, X_r^{t,x}, U_r(x), V_r(x)),$$

then for a.e.  $x \in \mathbb{R}^d$ , (2.6) becomes

$$Y_s^{t,x,n} = h(X_T^{t,x}) + \int_s^T \tilde{f}^x(r, Y_r^{t,x,n}, Z_r^{t,x,n}) dr - \sum_{j=1}^n \int_s^T \tilde{g}_j^x(r) d\hat{\beta}_j(r) - \int_s^T \langle Z_r^{t,x,n}, dW_r \rangle. \quad (2.12)$$

Then it is easy to see that for a.e.  $x \in \mathbb{R}^d$ ,  $\tilde{f}^x$  and  $\tilde{g}_j^x$  satisfy

(H.1)'.  $\tilde{f}^x : [t, T] \times \Omega \times \mathbb{R}^1 \times \mathbb{R}^d \rightarrow \mathbb{R}^1$  is  $\mathcal{B}_{[t,T]} \otimes \mathcal{F}_{s,T} \vee \mathcal{F}_{T,\infty}^{\hat{B}} \otimes \mathcal{B}_{\mathbb{R}^1} \otimes \mathcal{B}_{\mathbb{R}^d}$  measurable and

$\tilde{g}_j^x : [t, T] \times \Omega \rightarrow \mathbb{R}^1$  is  $\mathcal{B}_{[t,T]} \otimes \mathcal{F}_{s,T} \vee \mathcal{F}_{T,\infty}^{\hat{B}}$  measurable.

(H.2)'. For any  $r \in [t, T]$ ,  $y \in \mathbb{R}^1$ ,  $|\tilde{f}^x(r, y, z)| \leq M'_0(1 + |y| + |z|)$ .

(H.3)'. For any  $r \in [t, T]$ ,  $(y, z) \rightarrow \tilde{f}^x(r, y, z)$  is continuous.

By Theorem 4.1 in [22], for a.e.  $x \in \mathbb{R}^d$ , Eq.(2.12), as well as Eq.(2.6), has a solution  $(Y_s^{t,x,n}, Z_s^{t,x,n}) \in M^{2,0}([t, T]; \mathbb{R}^1) \otimes M^{2,0}([t, T]; \mathbb{R}^d)$ . In the following, we will prove that  $(Y_s^{t,x,n}, Z_s^{t,x,n}) \in M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$  under the conditions of Proposition 2.5.

First by Condition (H.4) or Condition (H.2)', Conditions (H.2), (H.3) and (H.7), for a.e.  $x \in \mathbb{R}^d$ , we have

$$E\left[\int_t^T |f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n})|^2 dr\right] + \sum_{j=1}^n E\left[\int_t^T |g_j(r, X_r^{t,x}, U_r(x), V_r(x))|^2 dr\right] < \infty.$$

Then for a.e.  $x \in \mathbb{R}^d$ , applying the generalized Itô's formula to  $e^{Kr}|Y_r^{t,x,n}|^2$ , we have

$$\begin{aligned} & E[e^{Ks}|Y_s^{t,x,n}|^2] + KE\left[\int_s^T e^{Kr}|Y_r^{t,x,n}|^2 dr\right] + E\left[\int_s^T e^{Kr}|Z_r^{t,x,n}|^2 dr\right] \\ &= E[e^{KT}|h(X_T^{t,x})|^2] + 2E\left[\int_s^T e^{Kr}Y_r^{t,x,n}f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n})dr\right] \\ & \quad + \sum_{j=1}^n E\left[\int_s^T e^{Kr}|g_j(r, X_r^{t,x}, U_r(x), V_r(x))|^2 dr\right]. \end{aligned}$$

Taking the integration over  $\mathbb{R}^d$  and by Conditions (H.1)–(H.5), (H.7) and Lemma 1.5, we have

$$\begin{aligned} & E\left[\int_{\mathbb{R}^d} e^{Ks}|Y_s^{t,x,n}|^2 \rho^{-1}(x)dx\right] + KE\left[\int_s^T \int_{\mathbb{R}^d} e^{Kr}|Y_r^{t,x,n}|^2 \rho^{-1}(x)dxdr\right] \\ & \quad + E\left[\int_s^T \int_{\mathbb{R}^d} e^{Kr}|Z_r^{t,x,n}|^2 \rho^{-1}(x)dxdr\right] \\ &= E\left[\int_{\mathbb{R}^d} e^{KT}|h(X_T^{t,x})|^2 \rho^{-1}(x)dx\right] + 2E\left[\int_s^T \int_{\mathbb{R}^d} e^{Kr}Y_r^{t,x,n}f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n})\rho^{-1}(x)dxdr\right] \\ & \quad + \sum_{j=1}^n E\left[\int_s^T \int_{\mathbb{R}^d} e^{Kr}|g_j(r, X_r^{t,x}, U_r(x), V_r(x))|^2 \rho^{-1}(x)dxdr\right] \\ &\leq C_p E\left[\int_{\mathbb{R}^d} |h(x)|^2 \rho^{-1}(x)dx\right] + (2\mu + 2C + 1)E\left[\int_s^T \int_{\mathbb{R}^d} e^{Kr}|Y_r^{t,x,n}|^2 \rho^{-1}(x)dxdr\right] \\ & \quad + \frac{1}{2}E\left[\int_s^T \int_{\mathbb{R}^d} e^{Kr}|Z_r^{t,x,n}|^2 \rho^{-1}(x)dxdr\right] + C_p + C_p E\left[\int_s^T \int_{\mathbb{R}^d} (|U_r(x)|^2 + |V_r(x)|^2)\rho^{-1}(x)dxdr\right] \\ & \quad + C_p \sum_{j=1}^n \int_s^T \int_{\mathbb{R}^d} |g_j(r, x, 0, 0)|^2 \rho^{-1}(x)dxdr. \end{aligned}$$

It turns out that

$$\begin{aligned} & E\left[\int_{\mathbb{R}^d} e^{Ks}|Y_s^{t,x,n}|^2 \rho^{-1}(x)dx\right] + (K - 2\mu - 2C - 1)E\left[\int_s^T \int_{\mathbb{R}^d} e^{Kr}|Y_r^{t,x,n}|^2 \rho^{-1}(x)dxdr\right] \\ & \quad + \frac{1}{2}E\left[\int_s^T \int_{\mathbb{R}^d} e^{Kr}|Z_r^{t,x,n}|^2 \rho^{-1}(x)dxdr\right] \\ &\leq C_p E\left[\int_{\mathbb{R}^d} |h(x)|^2 \rho^{-1}(x)dx\right] + C_p + C_p E\left[\int_s^T \int_{\mathbb{R}^d} (|U_r(x)|^2 + |V_r(x)|^2)\rho^{-1}(x)dxdr\right] \\ & \quad + C_p \sum_{j=1}^n \int_s^T \int_{\mathbb{R}^d} |g_j(r, x, 0, 0)|^2 \rho^{-1}(x)dxdr \\ &< \infty. \end{aligned} \tag{2.13}$$

Taking  $K$  sufficiently large, we can see that  $(Y^{\cdot, \cdot, n}, Z^{\cdot, \cdot, n}) \in M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$  and for a.e.  $x \in \mathbb{R}^d$ ,  $(Y_s^{t,x,n}, Z_s^{t,x,n})$  satisfies Eq.(2.6) on a full measure set  $\Omega^x \subset \Omega$

dependent on  $x$ . But we can use the Fubini theorem to perfect Eq.(2.6) so that  $(Y_s^{t,x,n}, Z_s^{t,x,n})$  satisfies (2.6) for a.e.  $x \in \mathbb{R}^d$  on a full measure set  $\tilde{\Omega}$  independent of  $x$ . To see this, from (2.13), we have for any  $s \in [t, T]$ ,

$$E\left[\int_{\mathbb{R}^d} e^{Ks} |Y_s^{t,x,n}|^2 \rho^{-1}(x) dx\right] = \int_{\mathbb{R}^d} E[e^{Ks} |Y_s^{t,x,n}|^2 \rho^{-1}(x)] dx < \infty, \quad (2.14)$$

so for a.e.  $x \in \mathbb{R}^d$ , there exists a full measure set  $\Omega^x \subset \Omega$  s.t.  $Y_s^{t,x,n} < \infty$  on  $\Omega^x$ . Denote the right hand side of (2.6) by  $F(s, x)$ . Then by Eq.(2.6), for  $x \in \mathbb{R}^d$ , there exists a full measure set  $\Omega'^x \subset \Omega$  s.t.  $Y_s^{t,x,n} = F(s, x)$  on  $\Omega'^x$ . Then for a.e.  $x \in \mathbb{R}^d$ , we have  $Y_s^{t,x,n} = F(s, x)$  on  $\Omega^x \cap \Omega'^x$ . Since now for a.e.  $x \in \mathbb{R}^d$ ,  $Y_s^{t,x,n} < \infty$  on  $\Omega^x \cap \Omega'^x$ , so  $F(s, x) < \infty$  and we can move  $F(s, x)$  to the other side of the equality to have  $Y_s^{t,x,n} - F(s, x) = 0$  on the full measure set  $\Omega^x \cap \Omega'^x$ . Thus

$$\int_{\mathbb{R}^d} E[|Y_s^{t,x,n} - F(s, x)|] dx = 0.$$

By the Fubini theorem, we have

$$E\left[\int_{\mathbb{R}^d} |Y_s^{t,x,n} - F(s, x)| dx\right] = 0.$$

This means that there exists a full measure set  $\dot{\Omega}$  independent of  $x$  s.t. on  $\dot{\Omega}$ ,  $Y_s^{t,x,n} - F(s, x) = 0$  for  $x \in \dot{\mathcal{E}}^\omega$ , where  $\dot{\mathcal{E}}^\omega$  is a full measure set in  $\mathbb{R}^d$  and depends on  $\omega$ . Similarly, from (2.14), we also know that there exists another full measure set  $\ddot{\Omega}$  independent of  $x$  s.t. on  $\ddot{\Omega}$ ,  $Y_s^{t,x,n} < \infty$  for  $x \in \ddot{\mathcal{E}}^\omega$ , where  $\ddot{\mathcal{E}}^\omega$  is a full measure set in  $\mathbb{R}^d$  and depends on  $\omega$ . Take  $\tilde{\Omega} = \dot{\Omega} \cap \ddot{\Omega}$  and  $\tilde{\mathcal{E}}^\omega = \dot{\mathcal{E}}^\omega \cap \ddot{\mathcal{E}}^\omega$ , then both are still a full measure set and on  $\tilde{\Omega}$ ,  $Y_s^{t,x,n} < \infty$  for  $x \in \tilde{\mathcal{E}}^\omega$ , furthermore  $F(s, x) < \infty$ . We can move items in the equality  $Y_s^{t,x,n} - F(s, x) = 0$  to have  $Y_s^{t,x,n} = F(s, x)$  for  $x \in \tilde{\mathcal{E}}^\omega$  on a full measure set  $\tilde{\Omega}$  independent of  $x$ .

Now we have  $(Y_s^{t,x,n}, Z_s^{t,x,n}) \in M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$  and for  $t \leq s \leq T$ ,  $(Y_s^{t,x,n}, Z_s^{t,x,n})$  satisfies (2.6) for a.e.  $x \in \mathbb{R}^d$  on a full measure set  $\tilde{\Omega}$  independent of  $x$ . Then for any  $\varphi \in C_c^0(\mathbb{R}^d; \mathbb{R}^1)$ , multiplying by  $\varphi$  on both sides of Eq.(2.6) and taking the integration over  $\mathbb{R}^d$ , we have that  $(Y_s^{t,x,n}, Z_s^{t,x,n})$  satisfies the spatial integral form of Eq.(2.6) for  $t \leq s \leq T$ . By Corollary 2.7,  $Y_r^{t,x,n} \in S^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$  and  $(Y_s^{t,x,n}, Z_s^{t,x,n})$  is a solution of Eq.(2.6).  $\diamond$

### 2.3 Existence and uniqueness of solutions of BDSDEs with infinite dimensional noise

Based on the results of BDSDEs with finite dimensional noise, now we can solve BDSDEs with infinite dimensional noise. Following a similar procedure as in the proof of Lemma 2.6, and applying Itô's formula to  $e^{Kr} |Y_r^{t,x,n}|^2$ , by the B-D-G inequality we first have the following estimation for the solution of Eq.(2.5):

**Proposition 2.8** *Under the conditions of Theorem 2.2,  $(Y_s^{t,x,n}, Z_s^{t,x,n})$  satisfies*

$$\sup_n E\left[\sup_{0 \leq s \leq T} \int_{\mathbb{R}^d} |Y_s^{t,x,n}|^2 \rho^{-1}(x) dx\right] + \sup_n E\left[\int_0^T \int_{\mathbb{R}^d} |Z_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right] < \infty.$$

Now we turn to the proof of the first main theorem of this section.

*Proof of Theorem 2.2.* The proof of the uniqueness is rather similar to the uniqueness proof in Theorem 2.4, so it is omitted.

Existence. By Theorem 2.4, for each  $n$ , there exists a unique solution  $(Y^{\cdot, \cdot, n}, Z^{\cdot, \cdot, n})$  to Eq.(2.5), so  $(Y^{\cdot, \cdot, n}, Z^{\cdot, \cdot, n}) \in S^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$  and for an arbitrary  $\varphi \in C_c^0(\mathbb{R}^d; \mathbb{R}^1)$ ,

$$\begin{aligned} \int_{\mathbb{R}^d} Y_s^{t,x,n} \varphi(x) dx &= \int_{\mathbb{R}^d} h(X_T^{t,x}) \varphi(x) dx + \int_s^T \int_{\mathbb{R}^d} f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) \varphi(x) dx dr \\ &\quad - \sum_{j=1}^n \int_s^T \int_{\mathbb{R}^d} g_j(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) \varphi(x) dx d\hat{\beta}_j(r) \\ &\quad - \int_s^T \left\langle \int_{\mathbb{R}^d} Z_r^{t,x,n} \varphi(x) dx, dW_r \right\rangle \quad P - \text{a.s.} \end{aligned} \quad (2.15)$$

We claim that  $(Y^{\cdot, \cdot, n}, Z^{\cdot, \cdot, n})$  is a Cauchy sequence in  $S^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$ . For this, applying Itô's formula to  $e^{Kr} |Y_r^{t,x,m} - Y_r^{t,x,n}|^2$  for a.e.  $x \in \mathbb{R}^d$ , we have

$$\begin{aligned} &\int_{\mathbb{R}^d} e^{Ks} |Y_s^{t,x,m} - Y_s^{t,x,n}|^2 \rho^{-1}(x) dx + \int_s^T \int_{\mathbb{R}^d} e^{Kr} |Y_r^{t,x,m} - Y_r^{t,x,n}|^2 \rho^{-1}(x) dx dr \\ &\quad + \int_s^T \int_{\mathbb{R}^d} e^{Kr} |Z_r^{t,x,m} - Z_r^{t,x,n}|^2 \rho^{-1}(x) dx dr \\ &\leq C_p \sum_{j=n+1}^m \{ (C_j + \alpha_j) \left( \int_s^T \int_{\mathbb{R}^d} (|Y_r^{t,x,m}|^2 + |Z_r^{t,x,m}|^2) \rho^{-1}(x) dx dr \right. \right. \\ &\quad \left. \left. + \int_s^T \int_{\mathbb{R}^d} |g_j(r, X_r^{t,x}, 0, 0)|^2 \rho^{-1}(x) dx dr \right) \right\} - \sum_{j=1}^n \int_s^T \int_{\mathbb{R}^d} 2e^{Kr} \bar{Y}_r^{t,x,m,n} \bar{g}_j^{m,n}(r, x) \rho^{-1}(x) dx d\hat{\beta}_j(r) \\ &\quad - \sum_{j=n+1}^m \int_s^T \int_{\mathbb{R}^d} 2e^{Kr} \bar{Y}_r^{t,x,m,n} g_j(r, X_r^{t,x}, Y_r^{t,x,m}, Z_r^{t,x,m}) \rho^{-1}(x) dx d\hat{\beta}_j(r) \\ &\quad - \int_s^T \left\langle \int_{\mathbb{R}^d} 2e^{Kr} \bar{Y}_r^{t,x,m,n} \bar{Z}_r^{t,x,m,n} \rho^{-1}(x) dx, dW_r \right\rangle. \end{aligned} \quad (2.16)$$

The claim is true by taking expectation and applying Lemma 1.5 and Proposition 2.8, as  $n, m \rightarrow \infty$

$$E \left[ \int_0^T \int_{\mathbb{R}^d} e^{Kr} |Y_r^{t,x,m} - Y_r^{t,x,n}|^2 \rho^{-1}(x) dx dr \right] + E \left[ \int_0^T \int_{\mathbb{R}^d} e^{Kr} |Z_r^{t,x,m} - Z_r^{t,x,n}|^2 \rho^{-1}(x) dx dr \right] \rightarrow 0 \quad (2.17)$$

and by the B-D-G inequality

$$E \left[ \sup_{0 \leq s \leq T} \int_{\mathbb{R}^d} e^{Ks} |Y_s^{t,x,m} - Y_s^{t,x,n}|^2 \rho^{-1}(x) dx \right] \rightarrow 0.$$

Denote its limit by  $(Y_s^{t,x}, Z_s^{t,x})$ .

We will show that  $(Y_s^{t,x}, Z_s^{t,x})$  satisfies (2.2) for an arbitrary  $\varphi \in C_c^0(\mathbb{R}^d; \mathbb{R}^1)$ . For this, we prove that along a subsequence (2.15), the spatial integral form of Eq.(2.5), converges to Eq.(2.2) in  $L^2(\Omega)$  term by term as  $n \rightarrow \infty$ . Here we only show that along a subsequence

$$E[|\int_s^T \int_{\mathbb{R}^d} (f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}))\varphi(x)dxdr|^2] \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Other items are under the same conditions as in Section 3 in [25], therefore the convergence can be dealt with similarly. Notice

$$\begin{aligned} & E[|\int_s^T \int_{\mathbb{R}^d} (f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}))\varphi(x)dxdr|^2] \\ & \leq TE[\int_s^T \int_{\mathbb{R}^d} |f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2 \rho^{-1}(x)dxdr \int_{\mathbb{R}^d} |\varphi(x)|^2 \rho(x)dx] \\ & \leq C_p E[\int_s^T \int_{\mathbb{R}^d} |f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2 \rho^{-1}(x)dxdr] \\ & \leq C_p E[\int_s^T \int_{\mathbb{R}^d} |f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) - f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x})|^2 \rho^{-1}(x)dxdr] \\ & \quad + C_p E[\int_s^T \int_{\mathbb{R}^d} |f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2 \rho^{-1}(x)dxdr] \\ & \leq C_p E[\int_s^T \int_{\mathbb{R}^d} |Z_r^{t,x,n} - Z_r^{t,x}|^2 \rho^{-1}(x)dxdr] \\ & \quad + C_p E[\int_s^T \int_{\mathbb{R}^d} |f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2 \rho^{-1}(x)dxdr]. \end{aligned}$$

We only need to prove that along a subsequence

$$E[\int_s^T \int_{\mathbb{R}^d} |f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2 \rho^{-1}(x)dxdr] \longrightarrow 0 \text{ as } n \longrightarrow \infty. \quad (2.18)$$

First we will find a subsequence of  $\{Y_r^{t,x,n}\}_{n=1}^\infty$  still denoted by  $\{Y_r^{t,x,n}\}_{n=1}^\infty$  s.t.  $Y_r^{t,x,n} \longrightarrow Y_r^{t,x}$  for a.e.  $r \in [0, T]$ ,  $x \in \mathbb{R}^d$ , a.s.  $\omega$  and  $E[\int_0^T \int_{\mathbb{R}^d} \sup_n |Y_r^{t,x,n}|^2 \rho^{-1}(x)dxdr] < \infty$ . For this, from (2.17), we know that  $E[\int_0^T \int_{\mathbb{R}^d} |Y_r^{t,x,n} - Y_r^{t,x}|^2 \rho^{-1}(x)dxdr] \longrightarrow 0$ . Therefore we may assume without losing any generality that  $Y_r^{t,x,n} \longrightarrow Y_r^{t,x}$  for a.e.  $r \in [0, T]$ ,  $x \in \mathbb{R}^d$ , a.s.  $\omega$  and extract a subsequence of  $\{Y_r^{t,x,n}\}_{n=1}^\infty$ , still denoted by  $\{Y_r^{t,x,n}\}_{n=1}^\infty$ , s.t.

$$\sqrt{E[\int_0^T \int_{\mathbb{R}^d} |Y_r^{t,x,n+1} - Y_r^{t,x,n}|^2 \rho^{-1}(x)dxdr]} \leq \frac{1}{2^n}.$$

For any  $n$ ,

$$|Y_r^{t,x,n}| \leq |Y_r^{t,x,1}| + \sum_{i=1}^{n-1} |Y_r^{t,x,i+1} - Y_r^{t,x,i}| \leq |Y_r^{t,x,1}| + \sum_{i=1}^{\infty} |Y_r^{t,x,i+1} - Y_r^{t,x,i}|.$$

Then by the triangle inequality of the norm, we have

$$\begin{aligned} & \sqrt{E[\int_0^T \int_{\mathbb{R}^d} \sup_n |Y_r^{t,x,n}|^2 \rho^{-1}(x)dxdr]} \\ & \leq \sqrt{E[\int_0^T \int_{\mathbb{R}^d} (|Y_r^{t,x,1}| + \sum_{i=1}^{\infty} |Y_r^{t,x,i+1} - Y_r^{t,x,i}|)^2 \rho^{-1}(x)dxdr]} \end{aligned}$$

$$\begin{aligned}
 &\leq \sqrt{E\left[\int_0^T \int_{\mathbb{R}^d} |Y_r^{t,x,1}|^2 \rho^{-1}(x) dx dr\right]} + \sum_{i=1}^{\infty} \sqrt{E\left[\int_0^T \int_{\mathbb{R}^d} |Y_r^{t,x,i+1} - Y_r^{t,x,i}|^2 \rho^{-1}(x) dx dr\right]} \\
 &\leq \sqrt{E\left[\int_0^T \int_{\mathbb{R}^d} |Y_r^{t,x,1}|^2 \rho^{-1}(x) dx dr\right]} + \sum_{i=1}^{\infty} \frac{1}{2^i} \\
 &< \infty.
 \end{aligned}$$

It therefore follows from Condition (H.4) that, for this subsequence  $\{Y_r^{t,x,n}\}_{n=1}^{\infty}$ ,

$$\begin{aligned}
 &E\left[\int_0^T \int_{\mathbb{R}^d} \sup_n |f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2 \rho^{-1}(x) dx dr\right] \\
 &\leq C_p E\left[\int_0^T \int_{\mathbb{R}^d} (1 + \sup_n |Y_r^{t,x,n}|^2 + |Y_r^{t,x}|^2 + |Z_r^{t,x}|^2) \rho^{-1}(x) dx dr\right] < \infty.
 \end{aligned}$$

Then, (2.18) follows from applying Lebesgue's dominated convergence theorem and Condition (H.6). The proof of Theorem 2.2 is complete.  $\diamond$

## 2.4 The corresponding SPDEs

We first consider the following SPDE with finite dimensional noise:

$$\begin{aligned}
 u^n(t, x) &= h(x) + \int_t^T [\mathcal{L}u^n(s, x) + f(s, x, u^n(s, x), (\sigma^* \nabla u^n)(s, x))] ds \\
 &\quad - \sum_{j=1}^n \int_t^T g_j(s, x, u^n(s, x), (\sigma^* \nabla u^n)(s, x)) d\hat{\beta}_j(s), \quad 0 \leq t \leq s \leq T.
 \end{aligned} \tag{2.19}$$

In the previous subsection, we proved the existence and uniqueness of solution of BDSDE (2.1) and obtained the solution  $(Y_s^{t,x}, Z_s^{t,x})$  by taking the limit of  $(Y_s^{t,x,n}, Z_s^{t,x,n})$  of the solutions of Eq.(2.5) in the space  $S^{2,0}([0, T]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([0, T]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^d))$  along a subsequence. We still start from Eq.(2.5) in this subsection.

**Proposition 2.9** *Under Conditions (H.1)–(H.7), assume Eq.(2.5) has a unique solution  $(Y_r^{t,x,n}, Z_r^{t,x,n})$ , then for any  $t \leq s \leq T$ ,*

$$Y_r^{s, X_s^{t,x}, n} = Y_r^{t,x,n} \text{ and } Z_r^{s, X_s^{t,x}, n} = Z_r^{t,x,n} \text{ for } r \in [s, T], \text{ a.a. } x \in \mathbb{R}^d \text{ a.s.}$$

*Proof.* The proof is similar to the proof of Proposition 3.4 in [25]. Here Lemma 2.6 plays the same role as Lemma 3.3 in that proof.  $\diamond$

A direct application of Proposition 2.9 and Fubini theorem immediately leads to

**Proposition 2.10** *Under Conditions (H.1)–(H.7), if we define  $u^n(t, x) = Y_t^{t,x,n}$ ,  $v^n(t, x) = Z_t^{t,x,n}$ , then*

$$u^n(s, X_s^{t,x}) = Y_s^{t,x,n}, \quad v^n(s, X_s^{t,x}) = Z_s^{t,x,n} \text{ for a.a. } s \in [t, T], \quad x \in \mathbb{R}^d \text{ a.s.}$$

**Theorem 2.11** *Under Conditions (H.1)–(H.7), if we define  $u^n(t, x) = Y_t^{t,x,n}$ , where  $(Y_s^{t,x,n}, Z_s^{t,x,n})$  is the solution of Eq.(2.5), then  $u^n(t, x)$  is the unique weak solution of Eq.(2.19). Moreover,*

$$u^n(s, X_s^{t,x}) = Y_s^{t,x,n}, \quad (\sigma^* \nabla u^n)(s, X_s^{t,x}) = Z_s^{t,x,n} \text{ for a.a. } s \in [t, T], \quad x \in \mathbb{R}^d \text{ a.s.}$$

*Proof. Uniqueness.* Let  $u^n$  be a solution of Eq.(2.19). Define

$$\begin{aligned} F^n(s, x) &= f(s, x, u^n(s, x), (\sigma^* \nabla u^n)(s, x)), \\ G_j^n(s, x) &= g_j(s, x, u^n(s, x), (\sigma^* \nabla u^n)(s, x)). \end{aligned}$$

Since  $u^n$  is a solution, so  $E[\int_0^T \int_{\mathbb{R}^d} (|u^n(s, x)|^2 + |(\sigma^* \nabla u^n)(s, x)|^2) \rho^{-1}(x) dx ds] < \infty$  and

$$\begin{aligned} & E[\int_0^T \int_{\mathbb{R}^d} (|F^n(s, x)|^2 + \sum_{j=1}^n |G_j^n(s, x)|^2) \rho^{-1}(x) dx ds] \\ &= E[\int_0^T \int_{\mathbb{R}^d} (|f(s, x, u^n(s, x), (\sigma^* \nabla u^n)(s, x))|^2 \\ &\quad + \sum_{j=1}^n |g_j(s, x, u^n(s, x), (\sigma^* \nabla u^n)(s, x))|^2) \rho^{-1}(x) dx ds] \\ &\leq C_p E[\int_0^T \int_{\mathbb{R}^d} (1 + |u^n(s, x)|^2 + |(\sigma^* \nabla u^n)(s, x)|^2 + \sum_{j=1}^n |g_j(s, x, 0, 0)|^2) \rho^{-1}(x) dx ds] \\ &< \infty. \end{aligned} \tag{2.20}$$

If we define  $Y_s^{t,x,n} = u^n(s, X_s^{t,x})$  and  $Z_s^{t,x,n} = (\sigma^* \nabla u^n)(s, X_s^{t,x})$ , then by Lemma 1.5,

$$\begin{aligned} & E[\int_t^T \int_{\mathbb{R}^d} (|Y_s^{t,x,n}|^2 + |Z_s^{t,x,n}|^2) \rho^{-1}(x) dx ds] \\ &\leq C_p E[\int_t^T \int_{\mathbb{R}^d} (|u^n(s, x)|^2 + |(\sigma^* \nabla u^n)(s, x)|^2) \rho^{-1}(x) dx ds] < \infty. \end{aligned}$$

Using some ideas of Theorem 2.1 in [3], similar to the argument as in Section 4 in [25], we have for  $t \leq s \leq T$ ,  $(Y_s^{t,\cdot,n}, Z_s^{t,\cdot,n}) \in M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$  solves the following BDSDE:

$$\begin{aligned} Y_s^{t,x,n} &= h(X_T^{t,x}) + \int_s^T F^n(r, X_r^{t,x}) dr \\ &\quad - \sum_{j=1}^n \int_s^T G_j^n(r, X_r^{t,x}) d^\dagger \hat{\beta}_j(r) - \int_s^T \langle Z_r^{t,x,n}, dW_r \rangle. \end{aligned} \tag{2.21}$$

Multiply  $\varphi \in C_c^0(\mathbb{R}^d; \mathbb{R}^1)$  on both sides and then take the integration over  $\mathbb{R}^d$ . Noting the definition of  $F^n(s, x)$ ,  $G_j^n(s, x)$ ,  $Y_s^{t,x,n}$  and  $Z_s^{t,x,n}$ , we have that  $(Y_s^{t,x,n}, Z_s^{t,x,n})$  satisfies the spatial integration form of Eq.(2.5). By Corollary 2.7,  $Y_s^{t,\cdot,n} \in S^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$  and therefore  $(Y_s^{t,x,n}, Z_s^{t,x,n})$  is a solution of Eq.(2.5). If there is another solution  $\hat{u}$  to Eq.(2.19), then by the same procedure, we can find another solution  $(\hat{Y}_s^{t,x,n}, \hat{Z}_s^{t,x,n})$  to Eq.(2.5), where

$$\hat{Y}_s^{t,x,n} = \hat{u}^n(s, X_s^{t,x}) \text{ and } \hat{Z}_s^{t,x,n} = (\sigma^* \nabla \hat{u}^n)(s, X_s^{t,x}).$$

By Theorem 2.4, the solution of Eq.(2.5) is unique, therefore

$$Y_s^{t,x,n} = \hat{Y}_s^{t,x,n} \text{ for a.a. } s \in [t, T], x \in \mathbb{R}^d \text{ a.s.}$$

Especially for  $t = 0$ ,

$$Y_s^{0,x,n} = \hat{Y}_s^{0,x,n} \text{ for a.a. } s \in [0, T], x \in \mathbb{R}^d \text{ a.s.}$$

By Lemma 1.5 again,

$$\begin{aligned} & E\left[\int_0^T \int_{\mathbb{R}^d} |u^n(s, x) - \hat{u}^n(s, x)|^2 \rho^{-1}(x) dx ds\right] \\ & \leq C_p E\left[\int_0^T \int_{\mathbb{R}^d} |u^n(s, X_s^{0,x}) - \hat{u}^n(s, X_s^{0,x})|^2 \rho^{-1}(x) dx ds\right] \\ & = C_p E\left[\int_0^T \int_{\mathbb{R}^d} |Y_s^{0,x,n} - \hat{Y}_s^{0,x,n}|^2 \rho^{-1}(x) dx ds\right] \\ & = 0. \end{aligned}$$

So  $u^n(s, x) = \hat{u}^n(s, x)$  for a.a.  $s \in [0, T]$ ,  $x \in \mathbb{R}^d$  a.s. Uniqueness is proved.

**Existence.** For each  $(t, x) \in [0, T] \otimes \mathbb{R}^d$ , define  $u^n(t, x) = Y_t^{t,x,n}$  and  $v^n(t, x) = Z_t^{t,x,n}$ , where  $(Y^{\cdot, \cdot, n}, Z^{\cdot, \cdot, n}) \in S^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$  is the solution of Eq.(2.5). Then by Proposition 2.10,

$$u^n(s, X_s^{t,x}) = Y_s^{t,x,n}, \quad v^n(s, X_s^{t,x}) = Z_s^{t,x,n} \text{ for a.a. } s \in [t, T], x \in \mathbb{R}^d \text{ a.s.}$$

Set

$$\begin{aligned} F^n(s, x) &= f(s, x, u^n(s, x), v^n(s, x)), \\ G_j^n(s, x) &= g_j(s, x, u^n(s, x), v^n(s, x)). \end{aligned}$$

Then it is easy to see that  $(Y^{\cdot, \cdot, n}, Z^{\cdot, \cdot, n}) \in M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$  is a solution of Eq.(2.21) with above  $F^n$  and  $G_j^n$ . Moreover, by Lemma 1.5,

$$E\left[\int_0^T \int_{\mathbb{R}^d} |u^n(s, x)|^2 + |v^n(s, x)|^2 \rho^{-1}(x) dx ds\right] < \infty.$$

Then from a similar computation as in (2.20) we have

$$E\left[\int_0^T \int_{\mathbb{R}^d} (|F^n(s, x)|^2 + \sum_{j=1}^n |G_j^n(s, x)|^2) \rho^{-1}(x) dx ds\right] < \infty.$$

Now using some ideas of Theorem 2.1 in [3], similar to the argument as in Section 4 in [25], we know that  $v^n(s, x) = (\sigma^* \nabla u^n)(s, x)$  and  $u^n$  is the weak solution of the following SPDE:

$$u^n(t, x) = h(x) + \int_t^T [\mathcal{L}u^n(s, x) + F^n(s, x)] ds - \sum_{j=1}^n \int_t^T G_j^n(s, x) d^\dagger \hat{\beta}_j(s), \quad 0 \leq t \leq s \leq T. \quad (2.22)$$

Noting that by the definition of  $F^n(s, x)$  and  $G_j^n(s, x)$  and the fact that  $v^n(s, x) = (\sigma^* \nabla u^n)(s, x)$ , from (2.22), we have that  $u^n$  is the weak solution of Eq.(2.19).  $\diamond$

In the rest part of this subsection, we study Eq.(1.3) with  $f$  and  $g$  allowed to depend on time. If  $(Y_s^{t,x}, Z_s^{t,x})$  is the solution of Eq.(2.1) and we define  $u(t, x) = Y_t^{t,x}$ , then by Proposition 4.2 in [25], we have  $\sigma^* \nabla u(t, x)$  exists for a.a.  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  a.s., and

$$u(s, X_s^{t,x}) = Y_s^{t,x}, (\sigma^* \nabla u)(s, X_s^{t,x}) = Z_s^{t,x} \text{ for a.a. } s \in [t, T], x \in \mathbb{R}^d \text{ a.s.} \quad (2.23)$$

Also by Theorem 2.11 and Lemma 1.5, we have

$$\begin{aligned} & E\left[\int_0^T \int_{\mathbb{R}^d} |u^n(s, x) - u(s, x)|^2 \rho^{-1}(x) dx ds\right] \\ & + E\left[\int_0^T \int_{\mathbb{R}^d} |(\sigma^* \nabla u^n)(s, x) - (\sigma^* \nabla u)(s, x)|^2 \rho^{-1}(x) dx ds\right] \\ & \leq C_p E\left[\int_0^T \int_{\mathbb{R}^d} |u^n(s, X_s^{0,x}) - u(s, X_s^{0,x})|^2 \rho^{-1}(x) dx ds\right] \\ & + C_p E\left[\int_0^T \int_{\mathbb{R}^d} |(\sigma^* \nabla u^n)(s, X_s^{0,x}) - (\sigma^* \nabla u)(s, X_s^{0,x})|^2 \rho^{-1}(x) dx ds\right] \\ & = C_p E\left[\int_0^T \int_{\mathbb{R}^d} |Y_s^{0,x,n} - Y_s^{0,x}|^2 \rho^{-1}(x) dx ds\right] \\ & + C_p E\left[\int_0^T \int_{\mathbb{R}^d} |Z_s^{0,x,n} - Z_s^{0,x}|^2 \rho^{-1}(x) dx ds\right] \longrightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (2.24)$$

With (2.24), we prove the other main theorem in this section.

*Proof of Theorem 2.3.* We only need to verify that this  $u$  defined through  $Y_t^{t,x}$  is the unique weak solution of Eq.(1.3). By Lemma 1.5 and (2.23), it is easy to see that

$$(\sigma^* \nabla u)(t, x) = Z_t^{t,x} \text{ for a.a. } t \in [0, T], x \in \mathbb{R}^d \text{ a.s.}$$

Furthermore, using the generalized equivalence norm principle again we have

$$\begin{aligned} & E\left[\int_0^T \int_{\mathbb{R}^d} (|u(s, x)|^2 + |(\sigma^* \nabla u)(s, x)|^2) \rho^{-1}(x) dx ds\right] \\ & \leq C_p E\left[\int_0^T \int_{\mathbb{R}^d} (|u(s, X_s^{0,x})|^2 + |(\sigma^* \nabla u)(s, X_s^{0,x})|^2) \rho^{-1}(x) dx ds\right] \\ & = C_p E\left[\int_0^T \int_{\mathbb{R}^d} (|Y_s^{0,x}|^2 + |Z_s^{0,x}|^2) \rho^{-1}(x) dx ds\right] < \infty. \end{aligned} \quad (2.25)$$

Now we will verify that  $u(t, x)$  satisfies (1.4). Since  $u^n(t, x)$  is the weak solution of SPDE (2.19), so for any  $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^1)$ ,  $u^n(t, x)$  satisfies

$$\begin{aligned} & \int_{\mathbb{R}^d} u^n(t, x) \varphi(x) dx - \int_{\mathbb{R}^d} h(x) \varphi(x) dx - \frac{1}{2} \int_t^T \int_{\mathbb{R}^d} (\sigma^* \nabla u^n)(s, x) (\sigma^* \nabla \varphi)(x) dx ds \\ & - \int_t^T \int_{\mathbb{R}^d} u^n(s, x) \nabla((b - \tilde{A})\varphi)(x) dx ds \\ & = \int_t^T \int_{\mathbb{R}^d} f(s, x, u^n(s, x), (\sigma^* \nabla u^n)(s, x)) \varphi(x) dx ds \\ & - \sum_{j=1}^n \int_t^T \int_{\mathbb{R}^d} g_j(s, x, u^n(s, x), (\sigma^* \nabla u^n)(s, x)) \varphi(x) dx d\hat{\beta}_j(s) \quad P - \text{a.s.} \end{aligned} \quad (2.26)$$

By proving that along a subsequence (2.26) converges to (1.4) in  $L^2(\Omega)$ , we have that  $u(t, x)$  satisfies (1.4). We only need to show that along a sequence as  $n \rightarrow \infty$ ,

$$E\left[\left|\int_t^T \int_{\mathbb{R}^d} (f(s, x, u^n(s, x), (\sigma^* \nabla u^n)(s, x)) - f(s, x, u(s, x), (\sigma^* \nabla u)(s, x))) \varphi(x) dx ds\right|^2\right] \longrightarrow 0.$$

First note

$$\begin{aligned} & E\left[\left|\int_t^T \int_{\mathbb{R}^d} (f(s, x, u^n(s, x), (\sigma^* \nabla u^n)(s, x)) - f(s, x, u(s, x), (\sigma^* \nabla u)(s, x))) \varphi(x) dx ds\right|^2\right] \\ & \leq C_p E\left[\int_t^T \int_{\mathbb{R}^d} |f(s, x, u^n(s, x), (\sigma^* \nabla u^n)(s, x)) - f(s, x, u(s, x), (\sigma^* \nabla u)(s, x))|^2 \rho^{-1}(x) dx ds\right] \\ & \leq C_p E\left[\int_t^T \int_{\mathbb{R}^d} |f(s, x, u^n(s, x), (\sigma^* \nabla u^n)(s, x)) - f(s, x, u^n(s, x), (\sigma^* \nabla u)(s, x))|^2 \rho^{-1}(x) dx ds\right] \\ & \quad + C_p E\left[\int_t^T \int_{\mathbb{R}^d} |f(s, x, u^n(s, x), (\sigma^* \nabla u)(s, x)) - f(s, x, u(s, x), (\sigma^* \nabla u)(s, x))|^2 \rho^{-1}(x) dx ds\right] \\ & \leq C_p E\left[\int_t^T \int_{\mathbb{R}^d} |(\sigma^* \nabla u^n)(s, x) - (\sigma^* \nabla u)(s, x)|^2 \rho^{-1}(x) dx ds\right] \\ & \quad + C_p E\left[\int_t^T \int_{\mathbb{R}^d} |f(s, x, u^n(s, x), (\sigma^* \nabla u)(s, x)) - f(s, x, u(s, x), (\sigma^* \nabla u)(s, x))|^2 \rho^{-1}(x) dx ds\right]. \end{aligned}$$

We face a similar situation as in (2.18) and only need to prove that along a subsequence as  $n \longrightarrow \infty$ ,

$$E\left[\int_t^T \int_{\mathbb{R}^d} |f(s, x, u^n(s, x), (\sigma^* \nabla u)(s, x)) - f(s, x, u(s, x), (\sigma^* \nabla u)(s, x))|^2 \rho^{-1}(x) dx ds\right] \longrightarrow 0. \quad (2.27)$$

For this, note that we have (2.24) which plays the same role as (2.17) in the proof of Theorem 2.2. Thus we can find a subsequence of  $\{u^n(s, x)\}_{n=1}^\infty$  still denoted by  $\{u^n(s, x)\}_{n=1}^\infty$  s.t.  $u^n(s, x) \longrightarrow u(s, x)$  for a.e.  $s \in [0, T]$ ,  $x \in \mathbb{R}^d$ , a.s.  $\omega$  and  $E[\int_0^T \int_{\mathbb{R}^d} \sup_n |u^n(s, x)|^2 \rho^{-1}(x) dx ds] < \infty$ . It turns out that, for this subsequence  $\{u^n(s, x)\}_{n=1}^\infty$ , by Condition (H.4), we have

$$E\left[\int_0^T \int_{\mathbb{R}^d} \sup_n |f(s, x, u^n(s, x), (\sigma^* \nabla u)(s, x)) - f(s, x, u(s, x), (\sigma^* \nabla u)(s, x))|^2 \rho^{-1}(x) dx ds\right] < \infty.$$

Thus (2.27) follows from using Lebesgue's dominated convergence theorem. Convergences of other terms in (2.26) are easy to check.

Therefore  $u(t, x)$  satisfies (1.4), i.e. it is a weak solution of Eq.(1.3) with  $u(T, x) = h(x)$ . We can prove the uniqueness following a similar argument in Theorem 2.11.  $\diamond$

### 3 Stationary Solutions of SPDEs and Infinite Horizon BDSDEs

In this section, first we will give the proof of Theorem 1.7. Then we show that the conditions in Theorem 1.7 are satisfied, i.e. both Theorem 1.8 and Theorem 1.9 are true under our assumptions.

#### 3.1 Proof of Theorem 1.7

*Proof.* First note that Eq.(1.7) is equivalent to the following BDSDE

$$\begin{cases} Y_s^{t,x} = Y_T^{t,x} + \int_s^T f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T g(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) d^\dagger \hat{B}_r - \int_s^T Z_r^{t,x} dW_r \\ \lim_{T \rightarrow \infty} e^{-KT} Y_T = 0 \quad \text{a.s.} \end{cases} \quad (3.1)$$

Let  $B_u = \hat{B}_{T'-u} - \hat{B}_{T'}$  for arbitrary  $T' > 0$  and  $-\infty < u \leq T'$ . Then  $B_u$  is a Brownian motion with  $B_0 = 0$ . For any  $r \geq 0$ , applying  $\hat{\theta}_r$  on  $B_u$ , we have

$$\begin{aligned} \hat{\theta}_r \circ B_u &= \hat{\theta}_r \circ (\hat{B}_{T'-u} - \hat{B}_{T'}) = \hat{B}_{T'-u+r} - \hat{B}_{T'+r} \\ &= (\hat{B}_{T'-u+r} - \hat{B}_{T'}) - (\hat{B}_{T'+r} - \hat{B}_{T'}) = B_{u-r} - B_{-r}. \end{aligned}$$

So for  $0 \leq s \leq T \leq T'$  and  $\{h(u, \cdot)\}_{u \geq 0}$  being a  $\mathcal{F}_u$ -measurable and locally square integrable stochastic process with values on  $\mathcal{L}_{U_0}^2(L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$ , we have the relationship between the forward integral and backward Itô integral (c.f. [25])

$$\int_s^T h(u, \cdot) d^\dagger B_u = - \int_{T'-T}^{T'-s} h(T'-u, \cdot) dB_u \quad \text{a.s.}$$

and for arbitrary  $T \geq 0$ ,  $0 \leq s \leq T$ ,  $r \geq 0$ ,

$$\hat{\theta}_r \circ \int_s^T h(u, \cdot) d^\dagger \hat{B}_u = \int_{s+r}^{T+r} \hat{\theta}_r \circ h(u-r, \cdot) d^\dagger \hat{B}_u. \quad (3.2)$$

Therefore for a.e.  $x \in \mathbb{R}^d$ ,

$$\hat{\theta}_r \circ \int_s^T h(u, x) d^\dagger \hat{B}_u = \int_{s+r}^{T+r} \hat{\theta}_r \circ h(u-r, x) d^\dagger \hat{B}_u.$$

Since  $(Y^{\cdot, \cdot}, Z^{\cdot, \cdot}) \in S^{2, -K} \cap M^{2, -K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2, -K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$  is the unique solution of Eq.(1.7), it follows that  $g(X^{\cdot, \cdot}, Y^{\cdot, \cdot}, Z^{\cdot, \cdot})$  is locally square integrable with values on  $\mathcal{L}_{U_0}^2(L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$ . Therefore by (1.6) and (3.2), for a.e.  $x \in \mathbb{R}^d$

$$\begin{aligned} \hat{\theta}_r \circ \int_s^T g(X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x}) d^\dagger \hat{B}_u &= \hat{\theta}_r \circ \int_{s+r}^{T+r} g(X_{u-r}^{t,x}, Y_{u-r}^{t,x}, Z_{u-r}^{t,x}) d^\dagger \hat{B}_u \\ &= \int_{s+r}^{T+r} g(X_u^{t+r,x}, \hat{\theta}_r \circ Y_{u-r}^{t,x}, \hat{\theta}_r \circ Z_{u-r}^{t,x}) d^\dagger \hat{B}_u. \end{aligned} \quad (3.3)$$

Now applying the operator  $\hat{\theta}_r$  on both sides of Eq.(3.1) and by (3.3), we know that  $\hat{\theta}_r \circ Y_s^{t,x}$  satisfies the following equation

$$\begin{cases} \hat{\theta}_r \circ Y_s^{t,x} = \hat{\theta}_r \circ Y_T^{t,x} + \int_{s+r}^{T+r} f(X_u^{t+r,x}, \hat{\theta}_r \circ Y_{u-r}^{t,x}, \hat{\theta}_r \circ Z_{u-r}^{t,x}) du \\ \quad - \int_{s+r}^{T+r} g(X_u^{t+r,x}, \hat{\theta}_r \circ Y_{u-r}^{t,x}, \hat{\theta}_r \circ Z_{u-r}^{t,x}) d^\dagger \hat{B}_u - \int_{s+r}^{T+r} \hat{\theta}_r \circ Z_{u-r}^{t,x} dW_u \\ \lim_{T \rightarrow \infty} e^{-K(T+r)} (\hat{\theta}_r \circ Y_T^{t,x}) = 0 \quad \text{a.s.} \end{cases} \quad (3.4)$$

On the other hand, from Eq.(3.1) it is obvious that

$$\begin{cases} Y_{s+r}^{t+r,x} = Y_{T+r}^{t+r,x} + \int_{s+r}^{T+r} f(X_u^{t+r,x}, Y_u^{t+r,x}, Z_u^{t+r,x}) du \\ \quad - \int_{s+r}^{T+r} g(X_u^{t+r,x}, Y_u^{t+r,x}, Z_u^{t+r,x}) d^\dagger \hat{B}_u - \int_{s+r}^{T+r} Z_u^{t+r,x} dW_u \\ \lim_{T \rightarrow \infty} e^{-K(T+r)} Y_{T+r}^{t+r,x} = 0 \quad \text{a.s.} \end{cases} \quad (3.5)$$

Let  $\hat{Y}^{t,\cdot} = \hat{\theta}_r \circ Y_{\cdot-r}^{t-r,\cdot}$ ,  $\hat{Z}^{t,\cdot} = \hat{\theta}_r \circ Z_{\cdot-r}^{t-r,\cdot}$ . By the uniqueness of the solution of Eq.(1.7) in the space  $S^{2, -K} \cap M^{2, -K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2, -K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$ , it follows from comparing (3.4) with (3.5) that for any  $r \geq 0$  and  $t \geq 0$ , in the space  $L_\rho^2(\mathbb{R}^d; \mathbb{R}^1) \otimes L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)$

$$\hat{\theta}_r \circ Y_s^{t,\cdot} = \hat{Y}_{s+r}^{t+r,\cdot} = Y_{s+r}^{t+r,\cdot}, \quad \hat{\theta}_r \circ Z_s^{t,\cdot} = \hat{Z}_{s+r}^{t+r,\cdot} = Z_{s+r}^{t+r,\cdot} \quad \text{for all } s \geq t \text{ a.s.}$$

Then by the perfection procedure ([1], [2]), we can prove above identities are true for all  $s \geq t$ ,  $r \geq 0$ , but fixed  $t \geq 0$  a.s. In particular, for any  $t \geq 0$ , in the space  $L_\rho^2(\mathbb{R}^d; \mathbb{R}^1) \otimes L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)$

$$\hat{\theta}_r \circ Y_t^{t,\cdot} = Y_{t+r}^{t+r,\cdot} \quad \text{for all } r \geq 0 \text{ a.s.} \quad (3.6)$$

From the assumptions, we also know that  $u(t, \cdot) \triangleq Y_t^{t,\cdot}$  is the continuous weak solution of Eq.(1.3). So we get from (3.6) that for any  $t \geq 0$ , in the space  $L_\rho^2(\mathbb{R}^d; \mathbb{R}^1) \otimes L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)$

$$\hat{\theta}_r \circ u(t, \cdot) = u(t+r, \cdot) \quad \text{for all } r \geq 0 \text{ a.s.}$$

Until now, we know "crude" stationary property for  $u(t, \cdot)$ , but due to the continuity of  $u(t, \cdot)$  w.r.t.  $t$  we can obtain an indistinguishable version of  $u(t, \cdot)$ , still denoted by  $u(t, \cdot)$ , s.t.

$$\hat{\theta}_r \circ u(t, \cdot) = u(t+r, \cdot) \quad \text{for all } t, r \geq 0 \text{ a.s.}$$

So we proved that  $u(t, \cdot)$  is a "perfect" stationary weak solution of Eq.(1.3).

By Definition 1.3, it follows that  $g(\cdot, u(s, \cdot), (\sigma^* \nabla u)(s, \cdot)) \in \mathcal{L}_{U_0}^2(L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$  should be locally square integrable. Now we consider Eq.(1.1) with cylindrical Brownian motion  $B$  on  $U_0$ . For arbitrary  $T' > 0$ , let  $Y$  be the solution of Eq.(1.7) and  $u(t, \cdot) = Y_t^{t,\cdot}$  be the stationary solution of Eq.(1.3) with  $\hat{B}$  chosen as the time reversal of  $B$  from time  $T'$ , i.e.  $\hat{B}_s = B_{T'-s} - B_{T'}$  for  $s \geq 0$ . Doing the integral transformation in the integration form (1.4) of Eq.(1.3), it is easy to see that  $v_t(x) \triangleq u(T' - t, x)$  satisfies (1.1).

In fact, we can prove a claim that  $v_t(\cdot)(\omega) = Y_{T'-t}^{T'-t,\cdot}(\hat{\omega})$  does not depend on the choice of  $T'$ . For this, we only need to show that for any  $T'^* \geq T'$ ,  $Y_{T'-t}^{T'-t,\cdot}(\hat{\omega}) = Y_{T'^*-t}^{T'^*-t,\cdot}(\hat{\omega}^*)$  when  $0 \leq t \leq T'$ , where  $\hat{\omega}(s) = B_{T'-s} - B_{T'}$  and  $\hat{\omega}^*(s) = B_{T'^*-s} - B_{T'^*}$ . Let  $\hat{\theta}$  and  $\hat{\theta}^*$  be the shifts of  $\hat{\omega}(\cdot)$  and  $\hat{\omega}^*(\cdot)$  respectively. Since by (3.6), we have

$$\begin{aligned} Y_{T'-t}^{T'-t,\cdot}(\hat{\omega}) &= \hat{\theta}_{T'-t} Y_0^{0,\cdot}(\hat{\omega}) = Y_0^{0,\cdot}(\hat{\theta}_{T'-t} \hat{\omega}), \\ Y_{T'^*-t}^{T'^*-t,\cdot}(\hat{\omega}^*) &= \hat{\theta}_{T'^*-t}^* Y_0^{0,\cdot}(\hat{\omega}^*) = Y_0^{0,\cdot}(\hat{\theta}_{T'^*-t}^* \hat{\omega}^*). \end{aligned}$$

So we only need to assert that  $\hat{\theta}_{T'-t} \hat{\omega} = \hat{\theta}_{T'^*-t}^* \hat{\omega}^*$ . Indeed we have for any  $s \geq 0$ ,

$$\begin{aligned} (\hat{\theta}_{T'-t} \hat{\omega})(s) &= \hat{\omega}(T' - t + s) - \hat{\omega}(T' - t) \\ &= (B_{T'-(T'-t+s)} - B_{T'}) - (B_{T'-(T'-t)} - B_{T'}) \\ &= B_{t-s} - B_t. \end{aligned}$$

Note that the right hand side of the above formula does not depend on  $T'$ , therefore  $\hat{\theta}_{T'-t} \hat{\omega}(s) = \hat{\theta}_{T'^*-t}^* \hat{\omega}^*(s) = B_{t-s} - B_t$ .

On the probability space  $(\Omega, \mathcal{F}, P)$ , we define  $\theta_t = (\hat{\theta}_t)^{-1}$ ,  $t \geq 0$ . Actually  $\hat{B}$  is a two-sided Brownian motion, so  $(\hat{\theta}_t)^{-1} = \hat{\theta}_{-t}$  is well defined (see [1]). It is easy to see that  $\theta_t$  is a shift w.r.t.  $B$  satisfying

- (i)  $P \cdot (\theta_t)^{-1} = P$ ;
- (ii)  $\theta_0 = I$ ;
- (iii)  $\theta_s \circ \theta_t = \theta_{s+t}$ ;
- (iv)  $\theta_t \circ B_s = B_{s+t} - B_t$ .

Since  $v_t(\cdot)(\omega) = u(T' - t, \cdot)(\hat{\omega}) = Y_{T'-t}^{T'-t, \cdot}(\hat{\omega})$  a.s., so

$$\theta_r v_t(\cdot)(\omega) = \hat{\theta}_{-r} u(T' - t, \cdot)(\hat{\omega}) = \hat{\theta}_{-r} \hat{\theta}_r u(T' - t - r, \cdot)(\hat{\omega}) = u(T' - t - r, \cdot)(\hat{\omega}) = v_{t+r}(\cdot)(\omega),$$

for all  $r \geq 0$  and  $T' \geq t + r$  a.s. In particular, let  $Y(\cdot)(\omega) = v_0(\cdot)(\omega) = Y_{T'}^{T', \cdot}(\hat{\omega})$ . Then the above formula implies:

$$\theta_t Y(\omega) = Y(\theta_t \omega) = v_t(\omega) = v(t, v_0(\omega), \omega) = v(t, Y(\omega), \omega) \text{ for all } t \geq 0 \text{ a.s.}$$

That is to say  $v_t(\cdot)(\omega) = v_0(\cdot)(\theta_t \omega) = Y(\cdot)(\theta_t \omega) = Y_{T'-t}^{T'-t, \cdot}(\hat{\omega})$  is a stationary solution of Eq.(1.1) w.r.t.  $\theta$ .  $\diamond$

### 3.2 The solution of infinite horizon BDSDE

We now consider the following infinite horizon BDSDE with infinite dimensional noise, which has a more general form than BDSDE (1.7):

$$\begin{aligned} e^{-Ks} Y_s^{t,x} &= \int_s^\infty e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_s^\infty K e^{-Kr} Y_r^{t,x} dr \\ &\quad - \int_s^\infty e^{-Kr} g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) d^\dagger \hat{B}_r - \int_s^\infty e^{-Kr} \langle Z_r^{t,x}, dW_r \rangle. \end{aligned} \quad (3.7)$$

Here  $f : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \rightarrow \mathbb{R}^1$ ,  $g : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \rightarrow \mathcal{L}_{U_0}^2(\mathbb{R}^1)$ . Eq.(3.7) is equivalent to

$$\begin{aligned} e^{-Ks} Y_s^{t,x} &= \int_s^\infty e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_s^\infty K e^{-Kr} Y_r^{t,x} dr \\ &\quad - \sum_{j=1}^\infty \int_s^\infty e^{-Kr} g_j(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) d^\dagger \hat{\beta}_j(r) - \int_s^\infty e^{-Kr} \langle Z_r^{t,x}, dW_r \rangle. \end{aligned}$$

We assume the previous conditions (H.2)-(H.6) with the following changes:

(H.8). Change “ $\mathcal{B}_{[0,T]}$ ” to “ $\mathcal{B}_{\mathbb{R}^+}$ ” and “ $r \in [0, T]$ ” to “ $r \geq 0$ ” in (H.2).

(H.9). Change “ $\int_0^T$ ” to “ $\int_0^\infty e^{-Kr}$ ” in (H.3).

(H.10). Change “ $r \in [0, T]$ ” to “ $r \geq 0$ ” in (H.4).

(H.11). Change “ $\mu \in \mathbb{R}^1$ ” to “ $\mu > 0$  with  $2\mu - K - 2C - \sum_{j=1}^\infty C_j > 0$ ”, “ $r \in [0, T]$ ” to “ $r \geq 0$ ” and “ $\leq \mu \int_{\mathbb{R}^d} U(x) |Y_1(x) - Y_2(x)|^2 \rho^{-1}(x) dx$ ” to “ $\leq -\mu \int_{\mathbb{R}^d} U(x) |Y_1(x) - Y_2(x)|^2 \rho^{-1}(x) dx$ ” in (H.5).

(H.12). Change “ $r \in [0, T]$ ” to “ $r \geq 0$ ” in (H.6).

Then we have the existence and uniqueness theorem for the general form BDSDE (3.7):

**Theorem 3.1** *Under Conditions (H.7)-(H.12), Eq.(3.7) has a unique solution.*

*Proof.* Here we only prove the existence of solutions as the uniqueness is similar to the procedure in the proof of uniqueness of Theorem 5.1 in [25] although we need the technique as in the proof of uniqueness of Theorem 2.4 to deal with the non-Lipschitz term. For each  $n \in \mathbb{N}$ , we define a sequence of BDSDEs by setting  $h = 0$  and  $T = n$  in Eq.(2.1):

$$\begin{aligned}
 Y_s^{t,x,n} &= \int_s^n f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) dr - \int_s^n g(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) d^\dagger \hat{B}_r \\
 &\quad - \int_s^n \langle Z_r^{t,x,n}, dW_r \rangle, \quad 0 \leq s \leq n.
 \end{aligned} \tag{3.8}$$

It is easy to verify that BDSDE (3.8) satisfies conditions of Theorem 2.2. Therefore, for each  $n$ , there exists  $(Y^{t,\cdot,n}, Z^{t,\cdot,n}) \in S^{2,-K}([0, n]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,-K}([0, n]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$  and  $(Y_s^{t,x,n}, Z_s^{t,x,n})$  is the unique solution of Eq.(3.8). That is to say, for an arbitrary  $\varphi \in C_c^0(\mathbb{R}^d; \mathbb{R}^1)$ ,  $(Y_s^{t,x,n}, Z_s^{t,x,n})$  satisfies

$$\begin{aligned}
 \int_{\mathbb{R}^d} e^{-Ks} Y_s^{t,x,n} \varphi(x) dx &= \int_s^n \int_{\mathbb{R}^d} e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) \varphi(x) dx dr \\
 &\quad + \int_s^n \int_{\mathbb{R}^d} K e^{-Kr} Y_r^{t,x,n} \varphi(x) dx dr \\
 &\quad - \sum_{j=1}^{\infty} \int_s^n \int_{\mathbb{R}^d} e^{-Kr} g_j(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) \varphi(x) dx d^\dagger \hat{\beta}_j(r) \\
 &\quad - \int_s^n \langle \int_{\mathbb{R}^d} e^{-Kr} Z_r^{t,x,n} \varphi(x) dx, dW_r \rangle \quad P - \text{a.s.}
 \end{aligned} \tag{3.9}$$

Let  $(Y_t^n, Z_t^n)_{t>n} = (0, 0)$ . Then  $(Y^{t,\cdot,n}, Z^{t,\cdot,n}) \in S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$ . Using a similar argument as in the proof of Theorem 5.1 in [25], we can prove that  $(Y_s^{t,x,n}, Z_s^{t,x,n})$  is a Cauchy sequence. Take  $(Y_s^{t,x}, Z_s^{t,x})$  as the limit of  $(Y_s^{t,x,n}, Z_s^{t,x,n})$  in the space  $S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$  and we will show that  $(Y_s^{t,x}, Z_s^{t,x})$  is a solution of Eq.(3.7). We only need to verify that for arbitrary  $\varphi \in C_c^0(\mathbb{R}^d; \mathbb{R}^1)$ ,  $(Y_s^{t,x}, Z_s^{t,x})$  satisfies

$$\begin{aligned}
 \int_{\mathbb{R}^d} e^{-Ks} Y_s^{t,x} \varphi(x) dx &= \int_s^\infty \int_{\mathbb{R}^d} e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \varphi(x) dx dr \\
 &\quad + \int_s^\infty \int_{\mathbb{R}^d} K e^{-Kr} Y_r^{t,x} \varphi(x) dx dr \\
 &\quad - \sum_{j=1}^{\infty} \int_s^\infty \int_{\mathbb{R}^d} e^{-Kr} g_j(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \varphi(x) dx d^\dagger \hat{\beta}_j(r) \\
 &\quad - \int_s^\infty \langle \int_{\mathbb{R}^d} e^{-Kr} Z_r^{t,x} \varphi(x) dx, dW_r \rangle \quad P - \text{a.s.}
 \end{aligned} \tag{3.10}$$

Noting that  $(Y_s^{t,x,n}, Z_s^{t,x,n})$  satisfies Eq.(3.9), we can prove that  $(Y_s^{t,x}, Z_s^{t,x})$  satisfies Eq.(3.10) by verifying that along a subsequence Eq.(3.9) converges to Eq.(3.10) in  $L^2(\Omega)$  term by term as  $n \rightarrow \infty$ . Here we only show that along a subsequence

$$\begin{aligned}
 E[ | \int_s^n \int_{\mathbb{R}^d} e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) \varphi(x) dx dr \\
 - \int_s^\infty \int_{\mathbb{R}^d} e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \varphi(x) dx dr |^2 ] \rightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}$$

For this, note that

$$E[ | \int_s^n \int_{\mathbb{R}^d} e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) \varphi(x) dx dr$$

$$\begin{aligned}
& - \int_s^\infty \int_{\mathbb{R}^d} e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \varphi(x) dx dr|^2] \\
\leq & 2E[|\int_s^n \int_{\mathbb{R}^d} e^{-Kr} (f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})) \varphi(x) dx dr|^2] \\
& + 2E[|\int_n^\infty \int_{\mathbb{R}^d} e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \varphi(x) dx dr|^2] \\
\leq & 2E[\int_s^\infty \int_{\mathbb{R}^d} e^{-Kr} |f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2 \rho^{-1}(x) dx dr] \\
& \times \int_s^\infty \int_{\mathbb{R}^d} e^{-Kr} |\varphi(x)|^2 \rho(x) dx dr \\
& + 2E[\int_n^\infty \int_{\mathbb{R}^d} e^{-Kr} |f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2 \rho^{-1}(x) dx dr] \int_n^\infty \int_{\mathbb{R}^d} e^{-Kr} |\varphi(x)|^2 \rho(x) dx dr \\
\leq & C_p E[\int_s^\infty \int_{\mathbb{R}^d} e^{-Kr} |f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2 \rho^{-1}(x) dx dr] \\
& + C_p E[\int_s^\infty \int_{\mathbb{R}^d} e^{-Kr} |f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2 \rho^{-1}(x) dx dr] \\
& + C_p E[\int_n^\infty \int_{\mathbb{R}^d} e^{-Kr} |f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2 \rho^{-1}(x) dx dr] \\
\leq & C_p E[\int_s^\infty \int_{\mathbb{R}^d} e^{-Kr} |Z_r^{t,x,n} - Z_r^{t,x}|^2 \rho^{-1}(x) dx dr] \\
& + C_p E[\int_n^\infty \int_{\mathbb{R}^d} e^{-Kr} (1 + |Y_r^{t,x}|^2 + |Z_r^{t,x}|^2) \rho^{-1}(x) dx dr] \\
& + C_p E[\int_s^\infty \int_{\mathbb{R}^d} e^{-Kr} |f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2 \rho^{-1}(x) dx dr].
\end{aligned}$$

Similar to (2.18), we only need to prove that along a subsequence

$$E[\int_s^\infty \int_{\mathbb{R}^d} e^{-Kr} |f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2 \rho^{-1}(x) dx dr] \longrightarrow 0 \text{ as } n \longrightarrow \infty. \quad (3.11)$$

Since  $\{Y_s^{t,x,n}\}_{n=1}^\infty$  is a Cauchy sequence in the space  $M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$  with the limit  $Y_s^{t,x}$ , as  $n \rightarrow \infty$ , we have

$$E[\int_0^\infty \int_{\mathbb{R}^d} e^{-Kr} |Y_r^{t,x,n} - Y_r^{t,x}|^2 \rho^{-1}(x) dx dr] \longrightarrow 0. \quad (3.12)$$

Then from (3.12) we can find a subsequence of  $\{Y_r^{t,x,n}\}_{n=1}^\infty$  still denoted by  $\{Y_r^{t,x,n}\}_{n=1}^\infty$  s.t.  $Y_r^{t,x,n} \rightarrow Y_r^{t,x}$  for a.e.  $r \geq 0, x \in \mathbb{R}^d$ , a.s.  $\omega$  and  $E[\int_0^\infty \int_{\mathbb{R}^d} e^{-Kr} \sup_n |Y_r^{t,x,n}|^2 \rho^{-1}(x) dx dr] < \infty$ . Therefore, for this subsequence  $\{Y_r^{t,x,n}\}_{n=1}^\infty$ , by Condition (H.10), we have

$$\begin{aligned}
& E[\int_0^\infty \int_{\mathbb{R}^d} e^{-Kr} \sup_n |f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2 \rho^{-1}(x) dx dr] \\
& \leq C_p E[\int_0^\infty \int_{\mathbb{R}^d} e^{-Kr} (1 + \sup_n |Y_r^{t,x,n}|^2 + |Y_r^{t,x}|^2 + |Z_r^{t,x}|^2) \rho^{-1}(x) dx dr] < \infty.
\end{aligned}$$

Then (3.11) follows from applying Lebesgue's dominated convergence theorem and Condition (H.12). That is to say  $(Y_s^{t,x}, Z_s^{t,x})_{s \geq 0}$  satisfies Eq.(3.10). The proof of Theorem 3.1 is complete.  $\diamond$

By a similar method as in the proof of the existence part in case (i) in Theorem 5.1 in [25], we have the following estimation:

**Proposition 3.2** *Let  $(Y_s^{t,x,n}, Z_s^{t,x,n})$  be the solution of Eq.(3.8). Then under the conditions of Theorem 3.1,*

$$\begin{aligned} & \sup_n E[\sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-Ks} |Y_s^{t,x,n}(x)|^2 \rho^{-1}(x) dx] + \sup_n E[\int_0^\infty \int_{\mathbb{R}^d} e^{-Kr} |Y_r^{t,x,n}(x)|^2 \rho^{-1}(x) dx dr] \\ & + \sup_n E[\int_0^\infty \int_{\mathbb{R}^d} e^{-Kr} |Z_r^{t,x,n}(x)|^2 \rho^{-1}(x) dx dr] < \infty. \end{aligned}$$

### 3.3 Proofs of Theorem 1.8 and Theorem 1.9

All the proofs until now in this paper have shown us how to deal with the non-Lipschitz term. Indeed the proofs of Theorem 1.8 and Theorem 1.9 are rather similar to the proofs in Section 6 in [25] even under the non-Lipschitz conditions. So we only intend to give the proof briefly.

*Proof of Theorem 1.8.* Since the conditions here are stronger than those in Theorem 3.1, so there exists a unique solution  $(Y_s^{t,x}, Z_s^{t,x})$  to Eq.(1.7). We only need to prove  $E[\sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} |Y_s^{t,x}|^p \rho^{-1}(x) dx] < \infty$ . Let  $\varphi_{N,p}(x) = x^{\frac{p}{2}} I_{\{0 \leq x < N\}} + N^{\frac{p-2}{2}} (\frac{p}{2}x - \frac{p-2}{2}N) I_{\{x \geq N\}}$ . We apply the generalized Itô's formula to  $e^{-pKr} \varphi_{N,p}(\psi_M(Y_r^{t,x}))$  for a.e.  $x \in \mathbb{R}^d$  to have

$$\begin{aligned} & e^{-pKs} \varphi_{N,p}(\psi_M(Y_s^{t,x})) - pK \int_s^T e^{-pKr} \varphi_{N,p}(\psi_M(Y_r^{t,x})) dr \\ & + \frac{1}{2} \int_s^T e^{-pKr} \varphi_{N,p}''(\psi_M(Y_r^{t,x})) |\psi_M'(Y_r^{t,x})|^2 |Z_r^{t,x}|^2 dr \\ & + \int_s^T e^{-pKr} \varphi_{N,p}'(\psi_M(Y_r^{t,x})) I_{\{-M \leq Y_r^{t,x} < M\}} |Z_r^{t,x}|^2 dr \\ & \leq e^{-pKT} \varphi_{N,p}(\psi_M(Y_T^{t,x})) + \int_s^T e^{-pKr} \varphi_{N,p}'(\psi_M(Y_r^{t,x})) \psi_M'(Y_r^{t,x}) f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr \\ & + \int_s^T e^{-pKr} \varphi_{N,p}'(\psi_M(Y_r^{t,x})) I_{\{-M \leq Y_r^{t,x} < M\}} \sum_{j=1}^\infty |g_j(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2 dr \\ & + \frac{1}{2} \int_s^T e^{-pKr} \varphi_{N,p}''(\psi_M(Y_r^{t,x})) |\psi_M'(Y_r^{t,x})|^2 \sum_{j=1}^\infty |g_j(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2 dr \\ & - \sum_{j=1}^\infty \int_s^T e^{-pKr} \varphi_{N,p}'(\psi_M(Y_r^{t,x})) \psi_M'(Y_r^{t,x}) g_j(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) d\hat{\beta}_j(r) \\ & - \int_s^T \langle e^{-pKr} \varphi_{N,p}'(\psi_M(Y_r^{t,x})) \psi_M'(Y_r^{t,x}) Z_r^{t,x}, dW_r \rangle. \end{aligned} \tag{3.13}$$

From the above estimation, using  $\lim_{T \rightarrow \infty} e^{-pKT} \varphi_{N,p}(\psi_M(Y_T^{t,x})) = 0$  and taking the limit as  $M \rightarrow \infty$  first, then the limit as  $N \rightarrow \infty$ , by the monotone convergence theorem, we have

$$\begin{aligned} & E[\int_s^\infty \int_{\mathbb{R}^d} e^{-pKr} |Y_r^{t,x}|^p \rho^{-1}(x) dx dr] + E[\int_s^\infty \int_{\mathbb{R}^d} e^{-pKr} |Y_r^{t,x}|^{p-2} |Z_r^{t,x}|^2 \rho^{-1}(x) dx dr] \\ & \leq C_p + C_p \int_{\mathbb{R}^d} \sum_{j=1}^\infty |g_j(x, 0, 0)|^p \rho^{-1}(x) dx < \infty. \end{aligned} \tag{3.14}$$

Also by the B-D-G inequality, the Cauchy-Schwartz inequality and the Young inequality, we can obtain another estimation from (3.13):

$$\begin{aligned} & E[\sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} |Y_s^{t,x}|^p \rho^{-1}(x) dx] \\ & \leq C_p \int_{\mathbb{R}^d} |f(x, 0, 0)|^p \rho^{-1}(x) dx + C_p \int_{\mathbb{R}^d} \sum_{j=1}^{\infty} |g_j(x, 0, 0)|^p \rho^{-1}(x) dx \\ & \quad + C_p E[\int_0^\infty \int_{\mathbb{R}^d} e^{-pKr} |Y_r^{t,x}|^{p-2} |Z_r^{t,x}|^2 \rho^{-1}(x) dx dr] + C_p E[\int_0^\infty \int_{\mathbb{R}^d} e^{-pKr} |Y_r^{t,x}|^p \rho^{-1}(x) dx dr]. \end{aligned}$$

So by (3.14), Theorem 1.8 is proved.  $\diamond$

Now we turn to the proof of Theorem 1.9.

*Proof of Theorem 1.9.* First note that we also can prove Lemma 6.2 in [25] under the conditions in this theorem, so we have

$$\begin{aligned} & E([\sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-2Ks} |Y_s^{t',x} - Y_s^{t,x}|^2 \rho^{-1}(x) dx])^{\frac{p}{2}} \\ & \leq C_p E[\sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKr} |Y_s^{t',x} - Y_s^{t,x}|^p \rho^{-1}(x) dx] (\int_{\mathbb{R}^d} \rho^{-1}(x) dx)^{\frac{p-2}{2}} \\ & \leq C_p |t' - t|^{\frac{p}{2}}. \end{aligned}$$

This is because we actually did not use the Lipschitz condition of  $f$  w.r.t.  $y$  and the monotone condition is enough. Noting  $p > 2$ , by the Kolmogorov continuity theorem (see [14]), we have  $t \rightarrow Y_s^{t,x}$  is a.s. continuous for  $t \in [0, T]$  under the norm  $(\sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-2Ks} |\cdot|^2 \rho^{-1}(x) dx)^{\frac{1}{2}}$ . Without losing any generality, assume that  $t' \geq t$ . Then we can see that

$$\lim_{t' \rightarrow t} (\int_{\mathbb{R}^d} e^{-2Kt'} |Y_{t'}^{t',x} - Y_{t'}^{t,x}|^2 \rho^{-1}(x) dx)^{\frac{1}{2}} \leq \lim_{t' \rightarrow t} (\sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-2Ks} |Y_s^{t',x} - Y_s^{t,x}|^2 \rho^{-1}(x) dx)^{\frac{1}{2}} = 0 \quad \text{a.s.}$$

Notice  $t' \in [0, T]$ , so

$$\lim_{t' \rightarrow t} (\int_{\mathbb{R}^d} |Y_{t'}^{t',x} - Y_{t'}^{t,x}|^2 \rho^{-1}(x) dx)^{\frac{1}{2}} = 0 \quad \text{a.s.} \quad (3.15)$$

Since  $Y_t^{\cdot, \cdot} \in S^{2, -K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$ ,  $Y_{t'}^{\cdot, \cdot}$  is continuous w.r.t.  $t'$  in  $L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)$ . That is to say for each  $t$ ,

$$\lim_{t' \rightarrow t} (\int_{\mathbb{R}^d} |Y_{t'}^{t,x} - Y_t^{t,x}|^2 \rho^{-1}(x) dx)^{\frac{1}{2}} = 0 \quad \text{a.s.} \quad (3.16)$$

Now by (3.15) and (3.16)

$$\begin{aligned} & \lim_{t' \rightarrow t} (\int_{\mathbb{R}^d} |Y_{t'}^{t',x} - Y_t^{t,x}|^2 \rho^{-1}(x) dx)^{\frac{1}{2}} \\ & \leq \lim_{t' \rightarrow t} (\int_{\mathbb{R}^d} |Y_{t'}^{t',x} - Y_{t'}^{t,x}|^2 \rho^{-1}(x) dx)^{\frac{1}{2}} + \lim_{t' \rightarrow t} (\int_{\mathbb{R}^d} |Y_{t'}^{t,x} - Y_t^{t,x}|^2 \rho^{-1}(x) dx)^{\frac{1}{2}} \\ & = 0 \quad \text{a.s.} \end{aligned}$$

For arbitrary  $T > 0$ ,  $0 \leq t \leq T$ , define  $u(t, \cdot) = Y_t^{t, \cdot}$ , then  $u(t, \cdot)$  is a.s. continuous w.r.t.  $t$  in  $L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)$ . Since  $Y_T^{\cdot, \cdot} \in S^{2, -K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$ ,  $Y_T^{T, x}$  is  $\mathcal{F}_{T, \infty}^{\hat{B}} \otimes \mathcal{B}_{\mathbb{R}^d}$  measurable and

$E[\int_{\mathbb{R}^d} |Y_T^{T,x}|^2 \rho^{-1}(x) dx] < \infty$ . It follows that Condition (H.1) is satisfied. Moreover, Conditions (A.1)–(A.6) are stronger than Conditions (H.2)–(H.7), so by Theorem 2.11,  $u(t, x)$  is a weak solution of Eq.(1.3). Theorem 1.9 is proved.  $\diamond$

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