Algebraic properties of compatible Poisson structures

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Abstract

To each pencil $C_\epsilon = A + \epsilon B$ of compatible Poisson brackets at a fixed point $x \in M$, we assign a linear algebraic group G_{C_ϵ} acting on T_xM and leaving both $A(x)$ and $B(x)$ invariant. We describe the corresponding Lie algebra $\mathfrak{g}_{C_\epsilon}$ and study its algebraic properties. The main result of the paper is an explicit formula for its dimension in terms of the so-called Kronecker–Jordan decomposition of the pencil of skew-symmetric forms generated by $A(x)$ and $B(x)$. We also give necessary and sufficient conditions for $\mathfrak{g}_{C_\epsilon}$ to be solvable and describe the semisimple Levi subalgebra $\mathfrak{h} \subset \mathfrak{g}_{C_\epsilon}$.

1 Introduction

In Hamiltonian mechanics [1, 2], a Poisson bracket [2, 3] on a smooth manifold M is defined to be a bilinear operation on the space of smooth functions, which satisfies three properties: skew symmetry, Jacobi identity and Leibnitz rule. There is a type of relation between Poisson brackets which is called *compatibility condition*: two Poisson brackets $\{\cdot,\cdot\}_A$ and $\{\cdot,\cdot\}_B$ are compatible if $\mu\{\cdot,\cdot\}_A + \lambda\{\cdot,\cdot\}_B$ is again a Poisson bracket (for any $\mu, \lambda \in \mathbb{R}$).

The 'bi-Hamiltonian' [4, 5, 6] property, i.e., the property of *being Hamiltonian w.r.t two different compatible Poisson structures*, is one of the most important mechanisms of integrability in Hamiltonian mechanics.

In differential geometry, when we study a certain structure, it is important to describe the transformation group that preserves the structure. The description of such a group for a pair of compatible Poisson brackets is one of the main goals of our research project.

This problem is rather complicated and we shall first discuss its linear version, which is the Lie group associated with compatible Poisson structures at a point.

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If we fix a point $x \in M$, then the compatible Poisson structures A and B can be viewed as skew-symmetric forms on a finite-dimensional vector space $V = T_x^* M$. Consider the pencil $C_\epsilon = B + \epsilon A$, $\epsilon \in \mathbb{R}$, of 2-forms generated by A and B. Then the group of linear transformations $X : V \to V$ that preserve this pencil is:

$$G_{C_\epsilon} = \{X \in GL(V) \mid C_\epsilon(v, u) = C_\epsilon(X(v), X(u)) \;\; \forall \, v, u \in V, \epsilon \in \mathbb{R}\}$$

The main difficulty in the description of this group is that its structure essentially depends on the properties of a given pencil. The Jordan-Kronecker decomposition theorem [5, 6, 11] (an analog of the Jordan decomposition theorem ([8] page 200) for linear operators) gives us the classification of all possible types of such pencils.

The main result of our work is an explicit formula for the dimension of this Lie group G_{C_ϵ} for each pencil $C_\epsilon = B + \epsilon A$ in terms of the sizes, number and types of blocks in the corresponding Jordan–Kronecker decomposition.

2 Basic notions and statement of the problem

Some basic definitions need to be introduced first.

Definition 2.1. [2, 3] In Hamiltonian mechanics, a Poisson bracket on a smooth manifold M is defined to be a bilinear operation on the space of smooth functions:

$$\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$$

which satisfies three properties:

- *skew symmetry:* $\{f, g\} = -\{g, f\}$*;*
- *Jacobi identity:* $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$*;*
- *Leibnitz rule:* $\{f, g_1 g_2\} = g_1\{f, g_2\} + g_2\{f, g_1\}$.

In local coordinates, each Poisson bracket is defined by:

$$\{f, g\} = \sum_{i,j} a^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} = (df)^\top A dg$$

where

$a^{ij}(x)$ is a skew-symmetric (2,0)-tensor; $df = \begin{pmatrix} \dfrac{\partial f}{\partial x^1} \\ \vdots \\ \dfrac{\partial f}{\partial x^n} \end{pmatrix}$; $dg = \begin{pmatrix} \dfrac{\partial g}{\partial x^1} \\ \vdots \\ \dfrac{\partial g}{\partial x^n} \end{pmatrix}$

As x is fixed, each Poisson bracket is given by a skew-symmetric matrix A.

A *Poisson manifold* is a smooth manifold with a Poisson bracket defined on its function space. A *Hamiltonian system* is a system of differential equations, and is usually formulated in terms of Hamiltonian vector fields on Poisson manifolds.

Definition 2.2. [1] To each function f on a Poisson manifold M, one can assign a Hamiltonian vector field ($f \to X_f$) in local coordinates:

$$X_f^j = \sum_i a^{ij}(x)\frac{\partial f}{\partial x^i} = Adf$$

In invariant terms, which means no coordinates are involved, if we think of a vector field as a derivation, then the notion of a Hamiltonian vector field leeds to a skew-symmetric bilinear operation on the differentiable functions (which is the Poisson bracket)

$$X_f(g) = \{g, f\}$$

There is a type of relation between Poisson brackets which is called *compatibility condition*:

Let $\{\cdot,\cdot\}_A$ and $\{\cdot,\cdot\}_B$ be two different Poisson brackets defined in local coordinates:

$$\{\cdot,\cdot\}_A = \sum_{i,j} a^{ij}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j} = (df)^\top Adg$$

$$\{\cdot,\cdot\}_B = \sum_{i,j} b^{ij}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j} = (df)^\top Bdg$$

Then these Poisson brackets $\{\cdot,\cdot\}_A$, $\{\cdot,\cdot\}_B$ are compatible if $\mu\{\cdot,\cdot\}_A + \lambda\{\cdot,\cdot\}_B$ is again a Poisson bracket (for any $\mu, \lambda \in \mathbb{R}$). Thus, two Poisson structures A and B are compatible if $\mu A + \lambda B$ is again a Poisson structure.

The 'bi-Hamiltonian' [4, 5, 6] property, i.e the property of being Hamiltonian w.r.t two different compatible Poisson structures, is one of the most important mechanisms of integrability in Hamiltonian mechanics.

In differential geometry, when we study a certain structure, it is important to describe the transformation group that preserves the structure. The description of such a group for a pair of compatible Poisson brackets is one of the main goals of our research project.

Let $\{\cdot,\cdot\}_A$ and $\{\cdot,\cdot\}_B$ be two compatible Poisson brackets related to a given bi-Hamiltonian system, and consider a set of the diffeomorphisms:

$$\Phi : M \longrightarrow M$$

that preserve the both Poisson brackets $\{\cdot,\cdot\}_A$ and $\{\cdot,\cdot\}_B$.

In other words,

$$\{f,g\}_A = \{\Phi \circ f, \Phi \circ g\}_A,$$
$$\{f,g\}_B = \{\Phi \circ f, \Phi \circ g\}_B.$$

We know the set of the diffeomorphisms preserving "something" is always a group, indeed a Lie group here, perhaps infinite dimensional.

This problem (about $\Phi : M \longrightarrow M$) is rather complicated and we shall first discuss its linear version, which is the Lie group (see [7] page 105) associated with compatible Poisson structures at a point.

Now, let us fix a point $x \in M$, and let A and B be two compatible Poisson structures viewed as skew-symmetric forms on a finite-dimensional vector space $V = T_x^* M$. We consider the pencil $C_\epsilon = B + \epsilon A$, $\epsilon \in \mathbb{R}$, of 2-forms generated by A and B.

Then the group of transformations $\tilde{X} : V \to V$ that preserve this pencil is:

$$G_{C_\epsilon} = \{\tilde{X} \in GL(V) \mid C_\epsilon(v,u) = C_\epsilon(\tilde{X}(v), \tilde{X}(u)) \ \ \forall\ v,u \in V, \epsilon \in \mathbb{R}\}$$

Since by the definition $\quad C_\epsilon(v,u) = \underline{v}^\top C_\epsilon \underline{u} \quad$ where $\quad \underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$

we have

$$\begin{aligned} \underline{v}^\top C_\epsilon \underline{u} &= (\tilde{X}\underline{v})^\top C_\epsilon (\tilde{X}\underline{u}) \\ &= \underline{v}^\top \tilde{X}^\top C_\epsilon \tilde{X} \underline{u} \end{aligned}$$

$$\Rightarrow \qquad \tilde{X}^\top C_\epsilon \tilde{X} = C_\epsilon.$$

Moreover G_{C_ϵ} is a Lie sub-group in $GL(n,\mathbb{R})$. Therefore in matrix notation, G_{C_ϵ} is defined by:

$$G_{C_\epsilon} = \{\tilde{X} \in GL(n,\mathbb{R}) \mid \tilde{X}^\top C_\epsilon \tilde{X} = C_\epsilon\}.$$

3 Jordan-Kronecker decomposition Theorem

Our aim is to describe the Lie group that preserves the pencil C_ϵ. Since the algebraic structure of this Lie group essentially depends on the properties of a given pencil, we need to consider the classification problem for pencils first.

<u>Classification problem at one point</u> [11]:

Given two pairs of skew-symmetric forms A_0, B_0 and A_1, B_1, we want to know whether or not there is a transformation $\phi : V \to V$ which sends A_0 to A_1 and B_0 to B_1.

In other words, we need to find out whether there exists an invertible matrix P s.t.

$$A_1 = P^\top A_0 P$$

$$B_1 = P^\top B_0 P$$

After the change of basis, we can see that A_1 and A_0 are congruent matrices, and so are B_1 and B_0. Two congruent matrices represent the same bilinear form w.r.t different bases. Obviously, for congruent pairs A_0, B_0 and A_1, B_1, the groups $G_{A_0+\epsilon B_0}$ and $G_{A_1+\epsilon B_1}$ are isomorphic.

The **Jordan-Kronecker decomposition Theorem** [4, 5, 11] gives us the answer to the Classification problem at one point.

Theorem 3.1. (Jordan–Kronecker decomposition theorem) *Let A and B be two skew-symmetric bilinear forms. Then by an appropriate choice of a basis, their matrices can be simultaneously reduced to the following canonical block-diagonal form:*

$$A \mapsto \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix} \qquad B \mapsto \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_k \end{pmatrix}$$

where the pairs of the corresponding blocks A_i and B_i can be of the following three types:

	A	B
Jordan block ($\lambda \in \mathbb{C}$)	$\begin{pmatrix} & J(\lambda) \\ -J^\top(\lambda) & \end{pmatrix}$	$\begin{pmatrix} & Id \\ -Id & \end{pmatrix}$
Jordan block ($\lambda = \infty$)	$\begin{pmatrix} & Id \\ -Id & \end{pmatrix}$	$\begin{pmatrix} & J(0) \\ -J^\top(0) & \end{pmatrix}$
Kronecker block	$\left(\begin{array}{c\|c} & \begin{matrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{matrix} \\ \hline \begin{matrix} -1 & & \\ 0 & \ddots & \\ & \ddots & -1 \\ & & 0 \end{matrix} & \end{array}\right)$	$\left(\begin{array}{c\|c} & \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{matrix} \\ \hline \begin{matrix} 0 & & \\ -1 & \ddots & \\ & \ddots & 0 \\ & & -1 \end{matrix} & \end{array}\right)$

Here λ plays the same role as an "eigenvalue" in the case of linear operators, when we consider the reduction to the standard Jordan normal form. More precisely, the $\lambda' s$ are those numbers for which the rank of $(B - \epsilon A)$ for $\epsilon = \lambda$ is not maximal. The case of Jordan blocks with $\lambda = \infty$ can always be avoided by replacing B with $B' = B + \mu A$ for a suitable μ. So from now on, we shall assume that ∞ is not a characteristic number, so that no Jordan block with "infinite eigenvalue" appears.

Therefore, instead of arbitrary skew-symmetric bilinear forms A and B, we can and shall describe the Lie group G_{C_ϵ} for standard canonical forms only.

From now on we consider the case when A and B are reduced to canonical forms.

There are 3 cases:

- Case(1) No Kronecker blocks (only Jordan blocks) or symplectic case;
- Case(2) No Jordan blocks (only Kronecker blocks) or degenerate case;
- Case(3) Mixed case (with both Jordan and Kronecker blocks as well as with zero blocks);

Instead of studying the Lie group that preserves both A and B, to make it easy in computation, we start with studying the corresponding Lie algebra (see [8] page 1, 202):

$$\mathfrak{g}_{C_\epsilon} = \{X \in gl(V) \mid C_\epsilon(X(v), u) + C_\epsilon(v, X(u)) = 0 \ \ \forall\, v, u \in V, \epsilon \in \mathbb{R}\}$$

In matrix notation:

$$\mathfrak{g}_{C_\epsilon} = \{X \in gl(n, \mathbb{R}) \mid X^\top C_\epsilon + C_\epsilon X = 0,\ \epsilon \in \mathbb{R}\}$$

Indeed, recall

$$G_{C_\epsilon} = \{\tilde{X} \in GL(n, \mathbb{R}) \mid \tilde{X}^\top C_\epsilon \tilde{X} = C_\epsilon\}$$

Let $\tilde{X} = \exp(tX)$ and differentiate the equation $(\exp(tX))^\top C_\epsilon \exp(tX)) = C_\epsilon$ with respect to t at $t = 0$. Then we have $X^\top C_\epsilon + C_\epsilon X = 0$, as needed.

For the reason of some convenience, we "transpose" this Lie algebra and consider the equivalent equation $XC_\epsilon + C_\epsilon X^\top = 0$ instead. The Lie algebra so obtained has, obviously, the same dimension. We shall still denote it by $\mathfrak{g}_{C_\epsilon}$. Notice that $\mathfrak{g}_{C_\epsilon} = \mathfrak{g}_A \cap \mathfrak{g}_B$.

The main result of our work is an explicit formula for the dimension of $\mathfrak{g}_{C_\epsilon}$, which is exactly the dimension of G_{C_ϵ} for each pencil $C_\epsilon = B + \epsilon A$ in the 3 cases described above.

Using the Jordan-Kronecker decomposition theorem we may assume without loss of generality that the pencil C_ϵ has been already reduced to a the canonical form

$$C_\epsilon = \begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_n \end{pmatrix}$$

Then the Lie algebra $\mathfrak{g}_{C_\epsilon}$ is determined by the (family of) matrix equations

$$XC_\epsilon + C_\epsilon X^\top = 0$$

which splits into several relatively simple matrix equations if we divide X into blocks in the following natural way

$$X = \begin{pmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ P_{21} & P_{22} & & \\ \vdots & & \ddots & \\ P_{n1} & & & P_{nn} \end{pmatrix}.$$

These equations are of the following form:

$$P_{ii}C_i + C_i(P_{ii})^\top = 0$$

or

$$\begin{cases} P_{ji}C_i + C_j(P_{ij})^\top = 0 \\ \\ P_{ij}C_j + C_i(P_{ji})^\top = 0 \end{cases}$$

where each C_i is either Kronecher, or Jordan, or just zero block.

It is important that the equations for different pairs of P_{ij}, P_{ji} are not linked, so we can solve them independently. As a result the dimension of the whole Lie algebra can be found as the sum of the dimensions of the spaces of solutions to each system of the above type.

The difficulty is, however, that the structure of the space of solutions essentially depends on C_i and C_j which can be both Kronecker, or both Jordan, or one Jordan and one Kronecker. Moreover, the sizes of blocks and the characteristic numbers (more precisely the fact whether or not they coincide) are very important. These cases will be treated in detail in the following sections.

In fact, we need to distinguish 9 essentially different cases. (The result for each case can be found in the Lemmas indicated below.)

- $i = j$ and C_i is a Kronecker block (refer to Lemma 4.1)
- $i = j$ and C_i is a Jordan block (refer to Lemma 5.1)

- $i = j$ and C_i is a zero block (refer to Lemma 7.2)
- C_i and C_j are both Kronecher but of different size (refer to Lemma 4.2)
- C_i and C_j are both Kronecker and of the same size (refer to Lemma 4.2)
- C_i and C_j are both Jordan with the same characteristic number (refer to Lemma 5.2)
- C_i and C_j are both Jordan but with distinct characteristic numbers (refer to Lemma 5.3)
- C_i is Jordan whereas C_j is Kronecker (refer to Lemma 6.1)
- C_i is zero, C_j is not (refer to Lemma 7.1)

4 Kronecker case

Bi-Hamiltonian systems related to compatible Poisson structure of pure Kronecker type appear in a very natural way in geometry and mathematical physics, see for example [4, 5, 6].

Theorem 4.1. *Consider a pencil $\mathcal{K}_\epsilon = A + \epsilon B$ of pure Kronecker type as below:*

$\mathcal{K}_\epsilon = \begin{pmatrix} \mathcal{K}_1 & & & \\ & \mathcal{K}_2 & & \\ & & \ddots & \\ & & & \mathcal{K}_q \end{pmatrix}$ *where $\mathcal{K}_i$ is an elementary Kronecker block of size $(2k_i + 1)$, i.e.*

$$\mathcal{K}_j = \begin{array}{c} \begin{array}{cc} \overbrace{\hphantom{-\epsilon \quad \ddots \quad \ddots \quad -\epsilon}}^{k_j} & \overbrace{\hphantom{\epsilon \quad 1 \quad \ddots \quad \ddots \quad \epsilon \quad 1}}^{k_j+1} \end{array} \\ \left(\begin{array}{cccc|ccccc} & & & & \epsilon & 1 & & & \\ & & & & & \ddots & \ddots & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & \epsilon & 1 \\ \hline -\epsilon & & & & & & & & \\ -1 & \ddots & & & & & & & \\ & \ddots & \ddots & & & & & & \\ & & \ddots & -\epsilon & & & & & \\ & & & -1 & & & & & \end{array}\right) \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} k_j \\ \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\} k_j+1 \end{array} \tag{1}$$

which is $(2k_j + 1) \times (2k_j + 1)$ square matrix. Assume that

1) $1 \le k_1 \le k_2 \le \cdots \le k_q$,
2) l is the number of blocks with distinct sizes,

3) m_i is the block multiplicity ($i = 1, 2, \cdots, l$), i.e,

$$\underbrace{\mathcal{K}_1 \cdots \mathcal{K}_{(m_1)}}_{\textit{with the same size}} \underbrace{\mathcal{K}_{(m_1+1)} \cdots \mathcal{K}_{m_1+m_2}}_{\textit{with the same size}} \cdots \underbrace{\mathcal{K}_{(q-m_l+1)} \cdots \mathcal{K}_q}_{\textit{with the same size}}$$

or, equavalently,

$$\underbrace{k_1 = \cdots = k_{m_1}}_{m_1} < \underbrace{k_{m_1+1} = \cdots = k_{m_1+m_2}}_{m_2} < \cdots < \underbrace{k_{q-m_l+1} \cdots k_q}_{m_l}.$$

Then the dimension of the Lie algebra $\mathfrak{g}_{\mathcal{K}_\epsilon}$ is:

$$\dim \mathfrak{g}_{\mathcal{K}_\epsilon} = \sum_{j=1}^{q} (2k_j + 1)j + \frac{1}{2} \sum_{i=1}^{l} m_i(m_i - 1)$$

Proof. (of the Dimension Theorem 4.1)
We consider $\mathfrak{g}_{\mathcal{K}_\epsilon}$ as a Lie subalgebra of $gl(n, \mathbb{R})$. In matrix notation, $\mathfrak{g}_{\mathcal{K}_\epsilon}$ is defined by one matrix equation:

$$\mathfrak{g}_{\mathcal{K}_\epsilon} = \{X \in gl(n, \mathbb{R}) | X\mathcal{K}_\epsilon + \mathcal{K}_\epsilon X^\top = 0\}$$

where ϵ is, however, an arbitrary parameter.

We write the elements of $\mathfrak{g}_{\mathcal{K}_\epsilon}$ in the form:

$$X = \left(\begin{array}{c|c|c|c} P_{11} & P_{12} & \dots & P_{1q} \\ \hline P_{21} & P_{22} & \dots & P_{2q} \\ \hline \vdots & \ddots & \vdots & \vdots \\ \hline P_{q1} & P_{q2} & \dots & P_{qq} \end{array}\right) \in \mathfrak{g}_{\mathcal{K}_\epsilon}$$

where P_{jj} is a square matrix of size $(2k_j + 1) \times (2k_j + 1)$. We also assume that $i < j$ when we use P_{ij} or P_{ji}.

By the definition of $\mathfrak{g}_{\mathcal{K}_\epsilon}$, it is easy to see that the sub-blocks in X satisfy the matrix equations:

$$\begin{array}{ll} \forall\ i = j & \qquad P_{jj}\mathcal{K}_j + \mathcal{K}_j(P_{jj})^\top = 0 \\ \forall\ i < j & \qquad P_{ji}\mathcal{K}_i + \mathcal{K}_j(P_{ij})^\top = 0 \end{array}$$

We denote by $\{P_{jj}\} \subset \mathfrak{g}_{\mathcal{K}_\epsilon}$ the subspace of matrices of the form: $\left(\begin{array}{c|c|c|c} 0 & 0 & 0 & \dots \\ \hline 0 & P_{jj} & 0 & \dots \\ \hline 0 & 0 & 0 & \dots \\ \hline \vdots & \vdots & \vdots & \ddots \end{array}\right)$,

similarly $\{P_{ij}\} \subset \mathfrak{g}_{\mathcal{K}_\epsilon}$ is the subspace of matrices: $\left(\begin{array}{c|c|c|c} 0 & P_{ij} & 0 & \dots \\ \hline P_{ji} & 0 & 0 & \dots \\ \hline 0 & 0 & 0 & \dots \\ \hline \vdots & \vdots & \vdots & \ddots \end{array}\right)$.

The proof we that we given is based on direct computation. We may break it down into a number of lemmas.

Lemma 4.1. (For $i = j$, $C_i = \mathcal{K}_j$ is a Kronecker block and $P_{ii} = P_{jj}$)

Consider the first matrix equation:

$$P_{jj}\mathcal{K}_j + \mathcal{K}_j(P_{jj})^\top = 0$$

where $\mathcal{K}_j$ is shown in (1).

Then the space of the solutions ($\{P_{jj}\}$) has dimension $2k_j + 1$, and the matrix P_{jj} has the following explicit form:

$$P_{jj} = \begin{array}{c} \begin{array}{cc} \overbrace{\qquad\qquad\qquad\quad}^{k_j} & \overbrace{\qquad\qquad\qquad\qquad\qquad\qquad}^{k_j+1} \end{array} \\ \left(\begin{array}{cccc|ccccc} a & & & & b_1 & b_2 & \cdots & b_{k_j} & b_{k_j+1} \\ & \ddots & & & b_2 & & \iddots & \iddots & \vdots \\ & & \ddots & & \vdots & \iddots & \iddots & & \vdots \\ & & & a & b_{k_j} & b_{k_j+1} & \cdots & \cdots & b_{2k_j} \\ \hline & & & & -a & & & & \\ & & & & & \ddots & & & \\ & & & & & & \ddots & & \\ & & & & & & & \ddots & \\ & & & & & & & & -a \end{array}\right) \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c} a \\ a \\ a \\ a \end{array}}\right\} k_j \\ \left.\vphantom{\begin{array}{c} a \\ a \\ a \\ a \\ a \end{array}}\right\} k_j + 1 \end{array}$$

The parameters $a, b_1, b_2, \cdots, b_{2k_j}$ within the matrix are arbitrary real (or complex) numbers.

Sketch of proof (Lemma 4.1):

Let $P_{jj} = \left(\begin{array}{c|c} X_1 & X_2 \\ \hline X_3 & X_4 \end{array}\right)$ be an unknown squared matrix, and denote $\mathcal{K}_j = \left(\begin{array}{c|c} & F \\ \hline -F^\top & \end{array}\right)$.

By using the matrix equation above, we obtain three independent equations:

$$\begin{aligned} X_1 F &= -F X_4^\top \\ X_2 F^\top &= F X_2^\top \\ X_3 F &= (X_3 F)^\top \end{aligned}$$

which can be easily solved separately, and the result is as shown above.

Lemma 4.2. (For $i < j$, $C_i = \mathcal{K}_i$ and $C_j = \mathcal{K}_j$ are both Kronecher)

Consider the second matrix equation:

$$P_{ji}\mathcal{K}_i + \mathcal{K}_j(P_{ji})^\top = 0 \qquad\qquad \forall\ i < j$$

where $\mathcal{K}_i$ and $\mathcal{K}_j$ are standard Kronecker blocks defined by (1) *of size $2k_i + 1$ and $2k_j + 1$ respectively.*

Then the matrices P_{ij} and P_{ji} have the following explicit form:

<u>*When $k_i < k_j$*</u>, ($C_i = \mathcal{K}_i$ and $C_j = \mathcal{K}_j$ are both Kronecher but of different size)

$$
P_{ij} = \begin{array}{cc}
\begin{array}{cc} \overbrace{\qquad\quad}^{k_j} & \overbrace{\qquad\qquad\qquad\qquad\qquad\qquad}^{k_j+1} \end{array} & \\
\left(\begin{array}{c|ccccc}
 & b_1 & b_2 & \cdots & b_{k_i}\cdots & b_{k_j+1} \\
 & b_2 & & \iddots & \iddots & \vdots \\
0 & \vdots & \iddots & \iddots & & \vdots \\
 & b_{k_i}\cdots & b_{k_j+1} & \cdots & \cdots & b_{k_i+k_j} \\
\hline
 & -c_1 & & & & \\
 & \vdots & \ddots & & & \\
 & -c_{k_j-k_i+1} & & \ddots & & \\
 & & \ddots & & \ddots & \\
0 & & & \ddots & & -c_1 \\
 & & & & \ddots & \vdots \\
 & & & & & -c_{k_j-k_i+1}
\end{array}\right)
&
\begin{array}{l}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} k_i \\
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\end{array}}\right\} k_i+1
\end{array}
\end{array}
$$

$$
P_{ji} = \begin{array}{cc}
\begin{array}{cc} \overbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad}^{k_i} & \overbrace{\qquad\qquad\qquad\qquad\qquad}^{k_i+1} \end{array} & \\
\left(\begin{array}{cccccc|ccccc}
c_1 & \cdots & c_{k_j-k_i+1} & & & & b_1 & b_2 & \cdots & b_{k_i} & b_{k_i+1} \\
 & \ddots & & \ddots & & & b_2 & & \iddots & \iddots & \vdots \\
 & & \ddots & & \ddots & & \vdots & \iddots & \iddots & & \vdots \\
 & & & c_1 & \cdots & c_{k_j-k_i+1} & b_{k_i} & b_{k_i+1} & \cdots & \cdots & b_{k_i+k_j} \\
\hline
 & & & & & & & & & & \\
 & & 0 & & & & & & 0 & & \\
 & & & & & & & & & &
\end{array}\right)
&
\begin{array}{l}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} k_j \\
\left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} k_j+1
\end{array}
\end{array}
$$

The number of independent parameters in the subspace $\{P_{ij}\}$ is $2k_j + 1$.

<u>*When $k_i = k_j$*</u>, ($C_i = \mathcal{K}_i$ and $C_j = \mathcal{K}_j$ are both Kronecker and of the same size)

$$
P_{ij} = \begin{array}{cc}
\begin{array}{cc} \overbrace{\qquad\qquad\quad}^{k_j} & \overbrace{\qquad\qquad\qquad\qquad\quad}^{k_j+1} \end{array} & \\
\left(\begin{array}{cccc|ccccc}
a & & & & b_1 & b_2 & \cdots & b_{k_j} & b_{k_j+1} \\
 & \ddots & & & b_2 & & \iddots & \iddots & \vdots \\
 & & \ddots & & \vdots & \iddots & \iddots & & \vdots \\
 & & & a & b_{k_j} & b_{k_j+1} & \cdots & \cdots & b_{2k_j} \\
\hline
 & & & & -c & & & & \\
 & & & & & \ddots & & & \\
 & & & & & & \ddots & & \\
 & & & & & & & \ddots & \\
 & & & & & & & & -c
\end{array}\right)
&
\begin{array}{l}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} k_j \\
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\} k_j+1
\end{array}
\end{array}
$$

$$
P_{ji} = \begin{array}{c}
\begin{array}{cc} \overbrace{\qquad\qquad\qquad}^{k_j} & \overbrace{\qquad\qquad\qquad\qquad\qquad\qquad}^{k_j+1} \end{array} \\
\left(\begin{array}{cccc|ccccc}
c & & & & b_1 & b_2 & \cdots & \cdots & b_{k_j+1} \\
& \ddots & & & b_2 & & \iddots & \iddots & \vdots \\
& & \ddots & & \vdots & \iddots & \iddots & & \vdots \\
& & & c & b_{k_j} & b_{k_j+1} & \cdots & \cdots & b_{2k_j} \\
\hline
& & & & -a & & & & \\
& & & & & \ddots & & & \\
& & & & & & \ddots & & \\
& & & & & & & \ddots & \\
& & & & & & & & -a
\end{array}\right)
\begin{array}{l}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} k_j \\
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\} k_j+1
\end{array}
\end{array}
$$

The number of independent parameters in the subspace $\{P_{ij}\}$ is $2k_j + 2$.

Proof. (Lemma 4.2):

Let $P_{ij} = \left(\begin{array}{c|c} Y_1 & Y_2 \\ \hline Y_3 & Y_4 \end{array}\right)$ and $P_{ji} = \left(\begin{array}{c|c} Z_1 & Z_2 \\ \hline Z_3 & Z_4 \end{array}\right)$ be unknown matrices.

Denote $\mathcal{K}_i = \left(\begin{array}{c|c} & F \\ \hline -F^\top & \end{array}\right)$, and $\mathcal{K}_j = \left(\begin{array}{c|c} & D \\ \hline -D^\top & \end{array}\right)$.

Applying the matrix equation above, for each case, we have 4 following independent equations:

For $k_i < k_j$ ($F \neq D$):

$$
\begin{aligned}
Y_1 D &= -F Z_4^\top & &\Rightarrow & &Y_1 = Z_4 = \{0\} \\
Y_2 D^\top &= F Z_2^\top & &\Rightarrow & &Y_2 \textit{ relates to } Z_2 \\
Y_3 D &= F^\top Z_3^\top & &\Rightarrow & &Y_3 = Z_3 = \{0\} \\
-Y_4 D^\top &= F^\top Z_1^\top & &\Rightarrow & &Y_4 \textit{ relates to } Z_1
\end{aligned}
$$

For $k_i = k_j$ ($F = D$):

$$
\begin{aligned}
Y_1 D &= -D Z_4^\top & &\Rightarrow & &Y_1 \textit{ relates to } Z_4 \\
Y_2 D^\top &= D Z_2^\top & &\Rightarrow & &Y_2 \textit{ relates to } Z_2 \\
Y_3 D &= D^\top Z_3^\top & &\Rightarrow & &Y_3 = Z_3 = \{0\} \\
-Y_4 D^\top &= D^\top Z_1^\top & &\Rightarrow & &Y_4 \textit{ relates to } Z_1
\end{aligned}
$$

Here we say that for example Y_1 relates to Z_4 in the sense that given any admissible Z_4 we can uniquely reconstruct Y_1. In particular, the number of independent parameters in Y_1 and Z_4 is the same. After the computations we may easily get the structures of P_{ij} and P_{ji} in

Lemma 4.2. We can see the space of solutions ($\{P_{ij}\}$) has dimension:

$$\begin{cases} 2k_j \;+\; 1, & if \quad k_i < k_j \\ 2k_j \;+\; 2, & if \quad k_i = k_j \end{cases}$$

$\square$

Back to the proof of Theorem 4.1, from Lemma 4.1 and Lemma 4.2, we have the following results.

First we consider the case when the size is strictly increasing, i.e. $k_1 < k_2 < \cdots < k_q$. According to our general scheme, the algebra $\mathfrak{g}_{\mathcal{K}_\epsilon}$ as a vector space is the direct sum of $\{P_{jj}\}$ and $\{P_{ij}\}$. In particular,

$$\dim \mathfrak{g}_{\mathcal{K}_\epsilon} = \sum_j \dim\{P_{jj}\} + \sum_{i<j} \dim\{P_{ij}\}.$$

Notice that the dimensions of two "block-subspaces" $\{P_{jl}\}$ and $\{P_{jm}\}$ are the same if $l, m < j$. We partition them into natural groups (rows) in the following way:

$$\begin{array}{l} \emptyset \\ \{P_{21}\} \\ \{P_{31}\}, \{P_{32}\} \\ \{P_{41}\}, \{P_{42}\}, \{P_{43}\} \\ \qquad \cdots \\ \{P_{q1}\}, \{P_{q2}\}, \ldots, \{P_{q(q-1)}\} \end{array}$$

The "block-subspaces" from the jth row all have the same dimension $2k_j + 1$ (see Lemma 4.2), and the number of blocks in the jth row is $j-1$. It gives us the dimension of $\oplus_{i<j}\{P_{ij}\}$, which is $\sum_{j=1}^{q}(2k_j + 1)(j - 1)$.

The dimension of each diagonal block which corresponds to the jth row (i.e. P_{jj}) is $2k_j + 1$ (see Lemma 4.1), so the sum of the dimensions of $\{P_{jj}\}$ is $\sum_{j=1}^{q}(2k_j + 1)$.

Then for $i < j$, $k_i < k_j$ the dimension obtained in this case is:

$$\begin{aligned} \dim \mathfrak{g}_{\mathcal{K}_\epsilon} &= \sum \dim\{P_{jj}\} + \sum \dim\{P_{ij}\} \\ &= \sum_{j=1}^{q}(2k_j + 1) + \sum_{j=1}^{q}(2k_j + 1)(j - 1) \\ &= \sum_{j=1}^{q}(2k_j + 1)j \end{aligned}$$

Next, we consider the case when the sizes of all blocks ($\mathcal{K}_i$) on the diagonal coincide.

Suppose there are m blocks (i.e. $\mathcal{K}_1, \ldots, \mathcal{K}_m$) on the diagonal of $\mathcal{K}_\epsilon$, and $k_1 = k_2 = \cdots = k_m = k$ and $k \geq 1$ in this case. Then each P_{jj} has dimension $2k+1$ as we mentioned before, thus the dimension of $\oplus_j^m \{P_{jj}\}$ is $m(2k+1)$.

Since $k_j = k$, each block in $\{P_{ij}\}$ has dimension $2k+2$ (see Lemma 4.2), and there are $\frac{m(m-1)}{2}$ independent blocks in $\oplus_{i<j}\{P_{ij}\}$.

Then for $i < j$, $k_i = k_j = k$ the dimension of this case is:

$$\begin{aligned}
\dim \mathfrak{g}_{\mathcal{K}_\epsilon} &= \sum \dim\{P_{jj}\} + \sum \dim\{P_{ij}\} \\
&= m(2k+1) + \frac{m(m-1)}{2} \cdot (2k+2) \\
&= m(2k+1) + m(m-1)(k+1) \\
&= m(2k+1) + m(mk + m - k - 1) \\
&= m(m + mk + k)
\end{aligned}$$

Combining the two cases above we have the mixture case:

Let the condition be $k_1 \leq k_2 \leq \cdots \leq k_q \ \forall j \in [1, 2, \cdots, q] \subset \mathbb{N}$ and $k_1 \geq 1$.
Same as before the dimension of diagonal blocks $(\oplus_j^q \{P_{jj}\})$ is $\sum_{j=1}^q (2k_j + 1)$, and the blocks from the ith row in the partition of $\{P_{ij}\}$ has dimension:

$$\begin{cases} 2k_j + 1, & \text{for } k_i < k_j \\ 2k_j + 2, & \text{for } k_i = k_j. \end{cases}$$

For every time when m multiplicity blocks appear together, for example:

$$k_1 < k_2 < \cdots < k_j < k_{j+1} = \cdots = k_{j+m} < \cdots < k_q$$

the total dimension of $\mathfrak{g}_{\mathcal{K}_\epsilon}$ increases by $\frac{m(m-1)}{2}$.
Finally we have,

$$\dim \mathfrak{g}_{\mathcal{K}_\epsilon} = \sum_{j=1}^{q} (2k_j + 1)j + \frac{1}{2} \sum_{i=1}^{l} m_i(m_i - 1).$$

□

5 Symplectic case

Non-degenerate compatible Poisson brackets are also related to many important problems in geometry and mathematical physics, see for example [2, 12, 13].

Theorem 5.1. *Let $\mathcal{J}_\epsilon = A + \epsilon B$ be a pencil of symplectic forms. Denote by $\lambda_1, \dots, \lambda_p$ all distinct characteristic numbers and let*

$$\begin{matrix} k_1(\lambda_1), \dots, k_{s_1}(\lambda_1), \\ k_1(\lambda_2), \dots, k_{s_2}(\lambda_2), \\ \cdots \\ k_1(\lambda_p), \dots, k_{s_p}(\lambda_p) \end{matrix} \qquad \textit{(where } k_1(\lambda_t) \geq k_2(\lambda_t) \geq \dots \ \forall\, t \in 1, \dots, p\textit{)}$$

be the sizes of the Jordan blocks associated with each eigenvalue.

Then the dimension of the Lie algebra $\mathfrak{g}_{\mathcal{J}_\epsilon}$ is

$$\sum_{t=1}^{p} \left(\sum_{j=1}^{s_t} (4j-1) \cdot k_j(\lambda_t) \right)$$

Proof. (of the Dimension Theorem 5.1)

In the Symplectic case, for $i \in 1, \cdots, n$ we have $\mathcal{J}_\epsilon$ in the Jordan block form:

$$\mathcal{J}_\epsilon = \begin{pmatrix} \mathcal{J}_1 & & \\ & \ddots & \\ & & \mathcal{J}_n \end{pmatrix} \text{ and } \mathcal{J}_i = \begin{array}{c} \begin{array}{cc} \overbrace{\hphantom{(-\lambda_t-\epsilon)\ \ -1\ \ \ddots\ \ (-\lambda_t-\epsilon)}}^{k_i} & \overbrace{\hphantom{(\lambda_t+\epsilon)\ \ 1\ \ \ddots\ \ (\lambda_t+\epsilon)}}^{k_i} \end{array} \\ \left(\begin{array}{cccc|cccc} & & & & (\lambda_t+\epsilon) & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \ddots & 1 \\ & & & & & & & (\lambda_t+\epsilon) \\ \hline (-\lambda_t-\epsilon) & & & & & & & \\ -1 & \ddots & & & & & & \\ & \ddots & \ddots & & & & & \\ & & -1 & (-\lambda_t-\epsilon) & & & & \end{array} \right) \begin{array}{l} \left.\vphantom{\begin{array}{c} a \\ a \\ a \\ a \end{array}}\right\} k_i \\ \left.\vphantom{\begin{array}{c} a \\ a \\ a \\ a \end{array}}\right\} k_i \end{array} \end{array}$$

The λ_t here is an arbitrary characteristic number (where $t \in 1, \dots, p$) w.r.t each block $\mathcal{J}_i$, which means some of characteristic numbers in different blocks ($\mathcal{J}_i$) may coincide, and some of them are not.

We have total p distinct characteristic numbers: $\lambda_1, \dots, \lambda_p$, and arrange the blocks ($\mathcal{J}_i$) with the same characteristic number next to each other on the diagonal and put them in the order from λ_1 to λ_p. So we have the order of $\mathcal{J}_i$ is like this:

$$\underbrace{\mathcal{J}_1 \dots \mathcal{J}_{s_1}}_{\lambda_1} \underbrace{\mathcal{J}_{(s_1+1)} \dots \mathcal{J}_{(s_1+s_2)}}_{\lambda_2} \cdots \underbrace{\mathcal{J}_{(n-s_p+1)} \dots \mathcal{J}_n}_{\lambda_p} \tag{2}$$

We use the same method as we did in the Kronecker case. First recall the matrix equation that defines $\mathfrak{g}_{\mathcal{J}_\epsilon}$:

$$\mathfrak{g}_{\mathcal{J}_\epsilon} = \{X \in gl(n, \mathbb{R}) \mid X\mathcal{J}_\epsilon + \mathcal{J}_\epsilon X^\top = 0\}$$

where ϵ is, however, an arbitrary parameter.

Then write the elements of $\mathfrak{g}_{\mathcal{J}_\epsilon}$ in the form:

$$X = \left(\begin{array}{c|c|c|c} P_{11} & P_{12} & \dots & P_{1n} \\ \hline P_{21} & P_{22} & \dots & P_{2n} \\ \hline \vdots & \ddots & \vdots & \vdots \\ \hline P_{n1} & P_{n2} & \dots & P_{nn} \end{array}\right) \in \mathfrak{g}_{\mathcal{J}_\epsilon}.$$

This time P_{ii} is a square matrix with the size $2k_i \times 2k_i$.

We also assume that $i < j$ when we use P_{ij} or P_{ji}.

By the definition of $\mathfrak{g}_{\mathcal{J}_\epsilon}$, it is easy to see that the sub-blocks in X satisfy the matrix equations:

$$\begin{array}{ll} \forall\ i = j & \qquad P_{ii}\mathcal{J}_i + \mathcal{J}_i(P_{ii})^\top = 0 \\ \forall\ i < j & \qquad P_{ji}\mathcal{J}_i + \mathcal{J}_j(P_{ij})^\top = 0 \end{array}$$

As we did in previous section, we denote by $\{P_{ii}\} \subset \mathfrak{g}_{\mathcal{J}_\epsilon}$ the subspace of matrices of the form: $\left(\begin{array}{c|c|c|c} 0 & 0 & 0 & \dots \\ \hline 0 & P_{ii} & 0 & \dots \\ \hline 0 & 0 & 0 & \dots \\ \hline \vdots & \vdots & \vdots & \ddots \end{array}\right)$, similarly $\{P_{ij}\} \subset \mathfrak{g}_{\mathcal{J}_\epsilon}$ is the subspace of matrices: $\left(\begin{array}{c|c|c|c} 0 & P_{ij} & 0 & \dots \\ \hline P_{ji} & 0 & 0 & \dots \\ \hline 0 & 0 & 0 & \dots \\ \hline \vdots & \vdots & \vdots & \ddots \end{array}\right)$.

Now we need the following three lemmas which can be proved just in the same way as the lemmas in the previous sections.

Lemma 5.1. (For $i = j$ and $C_i = \mathcal{J}_i$ is a Jordan block)

Let $P_{ii}\mathcal{J}_i + \mathcal{J}_i(P_{ii})^\top = 0$, where $\mathcal{J}_i$ is a Jordan block (as we denoted before). Then P_{ii} has the following explicit form:

$$P_{ii} = \begin{array}{c} \qquad \overbrace{\hphantom{a_1 \quad a_2 \quad \dots \quad a_{k_i}}}^{k_i} \quad \overbrace{\hphantom{b_{k_i} \quad \dots \quad b_2 \quad b_1}}^{k_i} \\ \left(\begin{array}{cccc|cccc} a_1 & a_2 & \dots & a_{k_i} & b_{k_i} & \dots & b_2 & b_1 \\ & \ddots & \ddots & \vdots & \vdots & ⋰ & ⋰ & \\ & & \ddots & a_2 & b_2 & ⋰ & & \\ & & & a_1 & b_1 & & & \\ \hline & & & c_1 & -a_1 & & & \\ & & ⋰ & c_2 & -a_2 & \ddots & & \\ & ⋰ & ⋰ & \vdots & \vdots & \ddots & \ddots & \\ c_1 & c_2 & \dots & c_{k_i} & -a_{k_i} & \dots & -a_2 & -a_1 \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c} a \\ a \\ a \\ a \end{array}}\right\} k_i \\ \left.\vphantom{\begin{array}{c} a \\ a \\ a \\ a \end{array}}\right\} k_i \end{array} \end{array}$$

and the elements $a_1, \cdots, a_{k_i},\ b_1, \cdots, b_{k_i},\ c_1, \cdots, c_{k_i}$ are independent arbitrary parameters. Thus, the number of independent parameters in $\{P_{ii}\}$ is $3k_i$.

Notice, that the block-subspace $\{P_{ii}\}$ as a Lie algebra is isomorphic to $sl(2,\mathbb{R})\otimes\mathbb{P}_{k_i-1}$, and can be reduced to the following matrix form by an appropriate change of a basis:

$$P_{ii} \simeq \begin{pmatrix} \boxed{\begin{matrix} a_1 & b_1 \\ c_1 & -a_1 \end{matrix}} & \boxed{\begin{matrix} a_2 & b_2 \\ c_2 & -a_2 \end{matrix}} & \cdots & \boxed{\begin{matrix} a_{k_i} & b_{k_i} \\ c_{k_i} & -a_{k_i} \end{matrix}} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \boxed{\begin{matrix} a_2 & b_2 \\ c_2 & -a_2 \end{matrix}} \\ & & & \boxed{\begin{matrix} a_1 & b_1 \\ c_1 & -a_1 \end{matrix}} \end{pmatrix} = sl(2,\mathbb{R})\otimes\mathbb{P}_{k_i-1},$$

where $\mathbb{P}_{k_i-1}$ denotes the algebra of truncated polynomials of degree k_i-1.

Lemma 5.2. (For $C_i=\mathcal{J}_i$ and $C_j=\mathcal{J}_j$ being both Jordan with the same characteristic number λ)

Let $P_{ji}\mathcal{J}_i+\mathcal{J}_j(P_{ij})^\top=0$, where $\mathcal{J}_i$ and $\mathcal{J}_j$ are Jordan blocks with the same characteristic number λ (here $i<j$ and $k_i\geq k_j$):

$$\mathcal{J}_i = \begin{array}{cc} \begin{array}{cc} \overbrace{\hphantom{(-\lambda-\epsilon)\quad -1 \quad (-\lambda-\epsilon)}}^{k_i} & \overbrace{\hphantom{(\lambda+\epsilon)\quad 1 \quad (\lambda+\epsilon)}}^{k_i} \end{array} & \\ \left(\begin{array}{cccc|cccc} & & & & (\lambda+\epsilon) & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \ddots & 1 \\ & & & & & & & (\lambda+\epsilon) \\ \hline (-\lambda-\epsilon) & & & & & & & \\ -1 & \ddots & & & & & & \\ & \ddots & \ddots & & & & & \\ & & -1 & (-\lambda-\epsilon) & & & & \end{array}\right) & \begin{array}{l} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} k_i \\ \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} k_i \end{array} \end{array}$$

$$\mathcal{J}_j = \begin{array}{cc} \begin{array}{cc} \overbrace{\hphantom{(-\lambda-\epsilon)\quad -1 \quad (-\lambda-\epsilon)}}^{k_j} & \overbrace{\hphantom{(\lambda+\epsilon)\quad 1 \quad (\lambda+\epsilon)}}^{k_j} \end{array} & \\ \left(\begin{array}{cccc|cccc} & & & & (\lambda+\epsilon) & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \ddots & 1 \\ & & & & & & & (\lambda+\epsilon) \\ \hline (-\lambda-\epsilon) & & & & & & & \\ -1 & \ddots & & & & & & \\ & \ddots & \ddots & & & & & \\ & & -1 & (-\lambda-\epsilon) & & & & \end{array}\right) & \begin{array}{l} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} k_j \\ \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} k_j \end{array} \end{array} .$$

Then we have

$$
P_{ij} = \begin{array}{cl}
\begin{array}{cc} \overbrace{\hphantom{d_1 \quad \dots \quad d_{k_j}}}^{k_j} & \overbrace{\hphantom{e_{k_j} \quad \dots \quad e_1}}^{k_j} \end{array} & \\
\left(\begin{array}{ccc|ccc}
d_1 & \dots & d_{k_j} & e_{k_j} & \dots & e_1 \\
0 & \ddots & \vdots & \vdots & \iddots & 0 \\
\vdots & & d_1 & e_1 & \iddots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 \\
\hline
0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \iddots & f_1 & -g_1 & \ddots & \vdots \\
0 & \iddots & \vdots & \vdots & \ddots & 0 \\
f_1 & \dots & f_{k_j} & -g_{k_j} & \dots & -g_1
\end{array}\right)
&
\begin{array}{l}
\left.\vphantom{\begin{array}{c} a \\ a \\ a \\ a \end{array}}\right\} k_i \\
\left.\vphantom{\begin{array}{c} a \\ a \\ a \\ a \end{array}}\right\} k_i
\end{array}
\end{array}
$$

and

$$
P_{ji} = \begin{array}{cl}
\begin{array}{cc} \overbrace{\hphantom{0 \quad g_1 \quad \dots \quad g_{k_j}}}^{k_i} & \overbrace{\hphantom{e_{k_j} \quad \dots \quad e_1 \quad 0}}^{k_i} \end{array} & \\
\left(\begin{array}{cccc|cccc}
0 & g_1 & \dots & g_{k_j} & e_{k_j} & \dots & e_1 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \iddots & \iddots & \vdots \\
0 & \dots & 0 & g_1 & e_1 & 0 & \dots & 0 \\
\hline
0 & \dots & 0 & f_1 & -d_1 & 0 & \dots & 0 \\
\vdots & \iddots & \iddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & f_1 & \dots & f_{k_j} & -d_{k_j} & \dots & -d_1 & 0
\end{array}\right)
&
\begin{array}{l}
\left.\vphantom{\begin{array}{c} a \\ a \\ a \end{array}}\right\} k_j \\
\left.\vphantom{\begin{array}{c} a \\ a \\ a \end{array}}\right\} k_j
\end{array}
\end{array}
\quad .
$$

There are $4k_j$ independent parameters in the subspace $\{P_{ij}\}$, i.e., $\dim\{P_{ij}\} = 4k_j$.

Lemma 5.3. (For $C_i = \mathcal{J}_i$ and $C_j = \mathcal{J}_j$ being both Jordan with distinct characteristic numbers $\lambda \neq \mu$)

Let $P_{ji}\mathcal{J}_i + \mathcal{J}_j(P_{ij})^\top = 0$, where $\mathcal{J}_i$ and $\mathcal{J}_j$ are Jordan blocks with different characteristic numbers λ and μ respectively, i.e.,

$$
\mathcal{J}_i = \left(\begin{array}{cccc|cccc}
 & & & & (\lambda+\epsilon) & 1 & & \\
 & & & & & \ddots & \ddots & \\
 & & & & & & \ddots & 1 \\
 & & & & & & & (\lambda+\epsilon) \\
\hline
(-\lambda-\epsilon) & & & & & & & \\
-1 & \ddots & & & & & & \\
 & \ddots & \ddots & & & & & \\
 & & -1 & (-\lambda-\epsilon) & & & &
\end{array}\right),
$$

$$
\mathcal{J}_j = \left(\begin{array}{cccc|cccc}
 & & & & (\mu+\epsilon) & 1 & & \\
 & & & & & \ddots & \ddots & \\
 & & & & & & \ddots & 1 \\
 & & & & & & & (\mu+\epsilon) \\
\hline
(-\mu-\epsilon) & & & & & & & \\
-1 & \ddots & & & & & & \\
 & \ddots & \ddots & & & & & \\
 & & -1 & (-\mu-\epsilon) & & & &
\end{array}\right).
$$

Then $P_{ij} = P_{ji} = \{0\}$.

After considering above three lemmas, we are now in the position to prove The Dimension Th 5.1, by working through these Lemmas. Lemma 5.3 tells us that $\{P_{ij}\} = \{0\}$ w.r.t $\mathcal{J}_i$ and $\mathcal{J}_j$ having distinct characteristic numbers. Thus if we arrange the blocks $(\mathcal{J}_i)$ in the above indicated way (2), then the Lie algebra $\mathfrak{g}_{\mathcal{J}_\epsilon}$ breaks down into independent diagonal blocks, i.e.,

$$X = \begin{pmatrix} \Lambda_1 & & 0 \\ & \ddots & \\ 0 & & \Lambda_p \end{pmatrix}$$

each of which splits into smaller sub-blocks:

$$\Lambda_t = \begin{pmatrix} P_{rr} & P_{r(r+1)} & \dots & P_{r(r+s_t)} \\ P_{(r+1)r} & P_{(r+1)(r+1)} & \dots & P_{(r+1)(r+s_t)} \\ \vdots & \vdots & \ddots & \vdots \\ P_{(r+s_t)r} & \dots & \dots & P_{(r+s_t)(r+s_t)} \end{pmatrix}$$

where $r = 1 + s_1 + \cdots + s_{(t-1)}$ $\forall \quad t \in 1, \dots, p$, and size $(P_{rr}) \geq$ size $(P_{(r+1)(r+1)}) \geq \cdots \geq$ size $(P_{(r+s_t)(r+s_t)}) \geq 2$. We relabel the size of each sub-block $(P_{(r+j)(r+i)})$ in Λ_t as $k_j(\lambda_t)$, so $k_1(\lambda_t) \geq k_2(\lambda_t) \geq \cdots \geq k_{s_t}(\lambda_t)$.

The blocks $\Lambda_1, \dots, \Lambda_p$ are "absolutely independent" in the sense that $\mathfrak{g}_{\mathcal{J}_\epsilon}$, as a vector space, is the direct sum of the block-subspaces $\{\Lambda_t\}$, and each Λ_t consists of non-zero $\{P_{(r+j)(r+j)}\}$ and $\{P_{(r+i)(r+j)}\}$ for $(r+i) < (r+j)$ and $i, j \in 0, \dots s_t$ (a similar notation as we defined $\{P_{jj}\}$ and $\{P_{ij}\}$). In other words, $\Lambda_t = \oplus_j \{P_{(r+j),(r+j)}\} + \oplus_{i,j} \{P_{(r+i),(r+j)}\}$.

We now follow the same scheme as for the Kronecker case in the previous section. Notice that in the subspace $\oplus_{i,j}\{P_{(r+i),(r+j)}\} \subset \Lambda_t$, the dimensions of two blocks $P_{(r+j)(r+l)}$ and $P_{(r+j)(r+m)}$ are the same if $j, l, m \in \{0, \dots s_t\}$ and $(r+l), (r+m) < (r+j)$. We do the same partition as we did in the Kronecker case:

$$\begin{array}{l} \emptyset \\ \{P_{(r+1)r}\} \\ \{P_{(r+2)r}\}, \{P_{(r+2)(r+1)}\} \\ \{P_{(r+3)r}\}, \{P_{(r+3)(r+1)}\}, \{P_{(r+3)(r+2)}\} \\ \qquad \dots \\ \{P_{(r+s_t)r}\}, \{P_{(r+s_t)(r+1)}\}, \dots, \{P_{(r+s_t)(r+s_t-1)}\} \end{array}$$

The "block-spaces" from the same $(r+j)th$ row all have the same dimension $4k_j$, which is given by the Lemma 5.2, and the number of blocks in the $(r+j)$th row is $j-1$. Therefore, $\sum_j^{s_t} \dim\{P_{(r+i)(r+j)}\} = \sum_{j=1}^{s_t} 4k_j(j-1)$.

However in the diagonal block subspace $\oplus_j\{P_{(r+j),(r+j)}\} \subset \Lambda_t$ for $j \in \{0, \dots, s_t\}$, Lemma 5.1 state the dimension of each $P_{(r+j)(r+j)}$ is $3k_j$. Relabeling k_j w.r.t each λ_t with $k_j(\lambda_t)$ as we mentioned before, we have $\sum_j^{s_t} \dim\{P_{(r+j)(r+j)}\} = \sum_{j=1}^{s+t} 3k_j(\lambda_t)$.

Thus,

$$
\begin{aligned}
\dim \mathfrak{g}_{\mathcal{J}_\epsilon} &= \sum_{t=1}^{p} (\dim\{\Lambda_t\}) \\
&= \sum_{t=1}^{p} \left(\sum_{j}^{s_t} \dim\{P_{(r+j)(r+j)}\} + \sum_{j}^{s_t} \dim\{P_{(r+j)(r+i)}\} \right) \\
&= \sum_{t=1}^{p} \left(\sum_{j=1}^{s_t} 3k_j + \sum_{j=1}^{s_t} 4k_j(j-1) \right) \\
&= \sum_{t=1}^{p} \left(\sum_{j=1}^{s_t} (4j-1)k_j \right) \\
&= \sum_{t=1}^{p} \left(\sum_{j=1}^{s_t} (4j-1) \cdot k_j(\lambda_t) \right),
\end{aligned}
$$

as was to be prooved. □

6 The mixture of Symplectic case and Kronecker blocks (with no zero blocks)

Theorem 6.1. *Let $\mathcal{C}_\epsilon = A + \epsilon B$ be a canonical pencil of skew-symmetric forms with the mixture of Jordan and Kronecker blocks:*

$$
\mathcal{C}_\epsilon = \left(\begin{array}{c|c} \mathcal{J}_\epsilon & \\ \hline & \mathcal{K}_\epsilon \end{array} \right)
$$

If we keep the same notation for $\mathcal{K}_\epsilon$ as in The Dimension Theorem 4.1 and for $\mathcal{J}_\epsilon$ as in The Dimension Theorem 5.1, then the dimension of the Lie algebra $\mathfrak{g}_{\mathcal{C}_\epsilon}$ is:

$$
\sum_{t=1}^{p} \left(\sum_{j=1}^{s_t} (2q+4j-1) \cdot k_j(\lambda_t) \right) + \sum_{j=1}^{q} (2k_j+1)j + \frac{1}{2} \sum_{i=1}^{l} m_i(m_i-1)
$$

where
1) $1 \leq k_1 \leq k_2 \leq \cdots \leq k_q$,
2) p is the number of distinct characteristic numbers,
3) $k_1(\lambda_t) \geq k_2(\lambda_t) \geq \cdots \geq k_{s_t}(\lambda_t)$ for any $t = 1, 2, \cdots, p$,
4) l is the number of Kronecker blocks with distinct sizes,
5) m_i is the Kronecker block multiplicity.

Proof. (of The Dimension Theorem 6.1)
The definition of $\mathfrak{g}_{C_\epsilon}$ state that

$$\mathfrak{g}_{\mathcal{C}_\epsilon} = \{X \in gl(n, \mathbb{R}) \mid X\mathcal{C}_\epsilon + \mathcal{C}_\epsilon X^\top = 0\}$$

Let us represent $\mathfrak{g}_{\mathcal{C}_\epsilon}$ in the form:

$$\mathfrak{g}_{\mathcal{C}_\epsilon} = \left\{ \left(\begin{array}{c|c} \mathfrak{g}_{11} & \mathfrak{g}_{12} \\ \hline \mathfrak{g}_{21} & \mathfrak{g}_{22} \end{array} \right) \right\}.$$

The partition is according to the type of different forms (i.e. Jordan and Kronecker forms).

By definition, we obtain the following equations:

$$\begin{aligned} \mathfrak{g}_{11}\mathcal{J}_\epsilon + \mathcal{J}_\epsilon(\mathfrak{g}_{11})^\top &= 0, \\ \mathfrak{g}_{22}\mathcal{K}_\epsilon + \mathcal{K}_\epsilon(\mathfrak{g}_{22})^\top &= 0. \end{aligned}$$

These give us the following:

$$\dim \mathfrak{g}_{11} = \sum_{t=1}^{p} \left(\sum_{j=1}^{s_t} (4j-1) \cdot k_j(\lambda_t) \right),$$

$$\dim \mathfrak{g}_{22} = \sum_{j=1}^{q} (2k_j + 1)j + \frac{1}{2} \sum_{i=1}^{l} m_i(m_i - 1).$$

We also have

$$\begin{aligned} \mathfrak{g}_{12}\mathcal{K}_\epsilon + \mathcal{J}_\epsilon(\mathfrak{g}_{21})^\top &= 0, \\ \mathfrak{g}_{21}\mathcal{J}_\epsilon + \mathcal{K}_\epsilon(\mathfrak{g}_{12})^\top &= 0, \end{aligned}$$

which implies that $\mathfrak{g}_{12}$ and $\mathfrak{g}_{21}$ are dependent on each other, so that $\dim\{\mathfrak{g}_{12}, \mathfrak{g}_{21}\} = \dim\{\mathfrak{g}_{12}\} = \dim\{\mathfrak{g}_{21}\}$.

The following Lemma gives the $\dim\{\mathfrak{g}_{12}, \mathfrak{g}_{21}\}$.

Lemma 6.1. ($C_i = \mathcal{J}_i$ is Jordan whereas $C_j = \mathcal{K}_j$ is Kronecker)

Let $P_{ij} \in \mathfrak{g}_{21}$, $P_{ji} \in \mathfrak{g}_{12}$, $C_i \in \mathcal{J}_\epsilon$ is Jordan and $C_j \in \mathcal{K}_\epsilon$ is Kronecker.

The solution of the equation $P_{ji}C_i + C_j(P_{ij})^\top = 0$ has the following explicit form:

$$P_{ij} = \left(\begin{array}{c|ccccc} 0 & \underline{a} & A\underline{a} & A^2\underline{a} & \dots & A^{k_i}\underline{a} \\ \hline 0 & -\underline{b} & -A^\top\underline{b} & -(A^\top)^2\underline{b} & \dots & (A^\top)^{k_i}\underline{a} \end{array} \right),$$

$$P_{ji} = \left(\begin{array}{c|c} \underline{b}^\top & \underline{a}^\top \\ \underline{b}^\top A & \underline{a}^\top A^\top \\ \underline{b}^\top A^2 & \underline{a}^\top (A^\top)^2 \\ \vdots & \vdots \\ \underline{b}^\top A^{k_i-1} & \underline{a}^\top (A^\top)^{k_i-1} \\ \hline 0 & 0 \end{array}\right)$$

$$\underline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_{k_i} \end{pmatrix} \ , \ \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_{k_i} \end{pmatrix} \ , \ A = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

The number of independent parameters in $\{P_{ij}\}$ is $2k_i$, which is exactly the size of the Jordan block.

The explicit form of P_{ij} and P_{ji} indicate that the $\dim\{\mathfrak{g}_{12}, \mathfrak{g}_{21}\}$ depends only on the size $(2k_i)$ of each Jordan block. If we arrange the size $(2k_i)$ in the certain order associated with distinct characteristic number as we described in the section 5, the $\dim\{\mathfrak{g}_{12}, \mathfrak{g}_{21}\}$ is then depending on $2k_j(\lambda_t)$. There are total of q Kronecker blocks with respect to each $2k_j(\lambda_t)$ in the subspace $\{\mathfrak{g}_{12}, \mathfrak{g}_{21}\}$. Sum up each $2qk_j(\lambda_t)$ according to the characteristic numbers should give us the dimension of the subspace $\{\mathfrak{g}_{12}, \mathfrak{g}_{21}\}$.

Thus,

$$\dim\{\mathfrak{g}_{12}, \mathfrak{g}_{21}\} = \sum_{t=1}^{p} \left(\sum_{j=1}^{s_t} 2qk_j(\lambda_t) \right)$$

Since $\{\mathfrak{g}_{11}\}$, $\{\mathfrak{g}_{22}\}$ and $\{\mathfrak{g}_{12}, \mathfrak{g}_{21}\}$ are pairwise independent, we have

$$\begin{aligned} \dim \mathfrak{g}_{\mathcal{C}_\epsilon} &= \dim \mathfrak{g}_{11} + \dim \mathfrak{g}_{22} + \dim\{\mathfrak{g}_{12}, \mathfrak{g}_{21}\} \\ &= \sum_{t=1}^{p} \left(\sum_{j=1}^{s_t} (4j-1) \cdot k_j(\lambda_t) \right) + \sum_{j=1}^{q} (2k_j+1)j + \frac{1}{2} \sum_{i=1}^{l} m_i(m_i-1) + 2q \sum_{t=1}^{p} \left(\sum_{j=1}^{s_t} k_j(\lambda_t) \right) \\ &= \sum_{t=1}^{p} \left(\sum_{j=1}^{s_t} (2q+4j-1) \cdot k_j(\lambda_i) \right) + \sum_{j=1}^{q} (2k_j+1)j + \frac{1}{2} \sum_{i=1}^{l} m_i(m_i-1) \end{aligned}$$

□

Here we give a simple example to clarify the situation.

Example:

Let

$$\mathcal{C}_\epsilon = \left(\begin{array}{c|c} \mathcal{J}_\epsilon & \\ \hline & \mathcal{K}_\epsilon \end{array}\right) = \left(\begin{array}{cccc|ccccc}
 & & \lambda+\epsilon & 1 & & & & & \\
 & & 0 & \lambda+\epsilon & & & & & \\
-\lambda-\epsilon & 0 & & & & & & & \\
-1 & -\lambda-\epsilon & & & & & & & \\ \hline
 & & & & & & \epsilon & 1 & \\
 & & & & & & 0 & \epsilon & 1 \\
 & & & & -\epsilon & 0 & & & \\
 & & & & -1 & -\epsilon & & & \\
 & & & & 0 & -1 & & &
\end{array}\right)$$

Then we obtain

$$P_{ij} = \left(\begin{array}{cc|ccc}
0 & 0 & a & \lambda a+b & \lambda(\lambda a+b)+\lambda b \\
0 & 0 & b & \lambda b & \lambda^2 b \\ \hline
0 & 0 & -c & -\lambda c & -\lambda^2 c \\
0 & 0 & -d & -\lambda d-c & \lambda(-\lambda d-c)-\lambda c
\end{array}\right), \quad P_{ji} = \left(\begin{array}{cc|cc}
c & d & a & b \\
\lambda c & \lambda d+c & \lambda a+b & \lambda b \\ \hline
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)$$

where

$$\underline{a} = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} c \\ d \end{pmatrix}, \quad A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

7 The mixed case involving zero blocks

Theorem 7.1. *Let $\mathcal{C}_\epsilon = B + \epsilon A$ be a canonical pencil of skew-symmetric forms with the mixture of Jordan, Kronecker and zero blocks:*

$$\mathcal{C}_\epsilon = \left.\left(\begin{array}{c|c|c} \mathcal{J}_\epsilon & & \\ \hline & \mathcal{K}_\epsilon & \\ \hline & & \overbrace{0}^{l_3} \end{array}\right)\right\} n$$

then

$$\dim \mathfrak{g}_{\mathcal{C}_\epsilon} = n \cdot l_3 + \sum_{t=1}^{p} \left(\sum_{j=1}^{s_t} (2q+4j-1) \cdot k_j(\lambda_i) \right) + \sum_{j=1}^{q} (2k_j+1)j + \frac{1}{2} \sum_{i=1}^{l} m_i(m_i-1).$$

Proof. Again, let us consider the following partition of $\mathfrak{g}_{\mathcal{C}_\epsilon}$:

$$\mathfrak{g}_{\mathcal{C}_\epsilon} = \left(\left(\begin{array}{c|c|c} \mathfrak{g}_{11} & \mathfrak{g}_{12} & \mathfrak{g}_{13} \\ \hline \mathfrak{g}_{21} & \mathfrak{g}_{22} & \mathfrak{g}_{23} \\ \hline \mathfrak{g}_{31} & \mathfrak{g}_{32} & \mathfrak{g}_{33} \end{array} \right) \right)$$

In section 6, we have already obtained the structure and dimension of the blocks: $\mathfrak{g}_{11}$, $\mathfrak{g}_{12}$, $\mathfrak{g}_{21}$,$\mathfrak{g}_{22}$.

Lemma 7.1. (For C_i is zero, C_j is not)

Let $P_{ij} \in \{\mathfrak{g}_{13}, \mathfrak{g}_{23}\}$, $P_{ji} \in \{\mathfrak{g}_{31}, \mathfrak{g}_{32}\}$, in particular, $P_{ji}C_i + C_j(P_{ij})^\top = 0$, where C_i is a zero block and C_j be either Jordan or Kronecker block. Then P_{ij} =arbitrary block and $P_{ji} = \{0\}$

Lemma 7.2. (For $i = j$ and C_i is a zero block)

Let $P_{ii} \in \mathfrak{g}_{33}$, i.e., $P_{ii}C_i + C_i(P_{ii})^\top = 0$ where C_i is just a zero block. Then P_{ii} is an arbitrary block.

Thus from these Lemmas we conclude that $\mathfrak{g}_{31}$ and $\mathfrak{g}_{32}$ contain only zero blocks, whereas $\mathfrak{g}_{13}$, $\mathfrak{g}_{23}$ and $\mathfrak{g}_{33}$ are blocks with arbitrary parameters. The dimension of $\{\mathfrak{g}_{13}, \mathfrak{g}_{23}, \mathfrak{g}_{33}\}$ is $n \cdot l_3$ and the dimension of $\{\mathfrak{g}_{31}, \mathfrak{g}_{32}\}$ is 0.

Therefore the complete dimension formula for $\mathfrak{g}_{\mathcal{C}_\epsilon}$ is:

$$\begin{aligned}\dim \mathfrak{g}_{\mathcal{C}_\epsilon} &= \dim\{\mathfrak{g}_{\mathcal{J}_\epsilon}\} + \dim\{\mathfrak{g}_{\mathcal{K}_\epsilon}\} + \dim\{\mathfrak{g}_{12}, \mathfrak{g}_{21}\} + \dim\{\mathfrak{g}_{13}, \mathfrak{g}_{23}, \mathfrak{g}_{33}\} + \dim\{\mathfrak{g}_{31}, \mathfrak{g}_{32}\} \\ &= \dim\{\mathfrak{g}_{\mathcal{J}_\epsilon}\} + \dim\{\mathfrak{g}_{\mathcal{K}_\epsilon}\} + \dim\{\mathfrak{g}_{12}, \mathfrak{g}_{21}\} + n \cdot l_3 \\ &= \sum_{t=1}^{p} \left(\sum_{j=1}^{s_t} (2q + 4j - 1) \cdot k_j(\lambda_i) \right) + \sum_{j=1}^{q} (2k_j + 1)j + \frac{1}{2} \sum_{i=1}^{l} m_i(m_i - 1) + n \cdot l_3,\end{aligned}$$

as was to be proved. $\square$

8 Some estimates for $\dim \mathfrak{g}_{C_\epsilon}$ and generic pencils

In the general case, the pencil C_ϵ is generated by three kinds of blocks: Jordan blocks, Kronecker blocks and zero blocks. According to different type of blocks we partition C_ϵ into three parts on the diagonal: Jordan part ($\mathcal{J}_\epsilon$), Kronecker part ($\mathcal{K}_\epsilon$) and zero part (0), as shown below.

$$C_\epsilon = \begin{array}{c} \begin{array}{ccc} \overbrace{\quad}^{l_1} & \overbrace{\quad}^{l_2} & \overbrace{\quad}^{l_3} \end{array} \\ \left(\begin{array}{c|c|c} \mathcal{J}_\epsilon & & \\ \hline & \mathcal{K}_\epsilon & \\ \hline & & 0 \end{array}\right) \end{array} \begin{array}{l} \\ \} \, l_1 \\ \} \, l_2 \\ \} \, l_3 \end{array}$$

$$\mathfrak{g}_{C_\epsilon} = \begin{array}{c} \begin{array}{ccc} \overbrace{\qquad}^{l_1} & \overbrace{\qquad}^{l_2} & \overbrace{\qquad\qquad}^{l_3} \end{array} \\ \left(\begin{array}{c|c|c} \mathfrak{g}_{\mathcal{J}_\epsilon} & P_{ij} & \text{Arbitrary} \\ \hline P_{ji} & \mathfrak{g}_{\mathcal{K}_\epsilon} & \text{Arbitrary} \\ \hline 0 & 0 & \text{Arbitrary} \end{array}\right) \end{array} \begin{array}{l} \\ \} \, l_1 \\ \} \, l_2 \\ \} \, l_3 \end{array}$$

Notice, C_ϵ is $n \times n$ matrix and $l_1 + l_2 + l_3 = n \geq 1$.

Based on the previous discussions with respect to each kind of blocks, we can easily obtain the following estimates for $\dim \mathfrak{g}_{C_\epsilon}$.

Lemma 8.1. $\dim \mathfrak{g}_{\mathcal{J}_\epsilon} \geq \frac{3l_1}{2}$.

Proof. According to Theorem 5.1, we have

$$\begin{aligned}
\dim \mathfrak{g}_{\mathcal{J}_\epsilon} &= \sum_{t=1}^{p}\left(\sum_{j=1}^{s_t}(4j-1)\cdot k_j(\lambda_t)\right)\\
(\text{Since } j \geq 1, \text{then } (4j-1)\geq 3) \qquad &\geq \sum_{t=1}^{p}\left(\sum_{j=1}^{s_t}3k_j(\lambda_t)\right)\\
&= 3\sum_{t=1}^{p}\left(\sum_{j=1}^{s_t}k_j(\lambda_t)\right)\\
&= 3\cdot(\text{sum of all } k_j \text{ over all } \lambda_t)\\
&= 3\cdot(\text{half of the size of the Jordan part})\\
&= \frac{3l_1}{2}
\end{aligned}$$

□

Lemma 8.2. $\dim \mathfrak{g}_{\mathcal{K}_\epsilon} \geq l_2$.

Proof. According to Theorem 4.1,

$$\begin{aligned}
\dim \mathfrak{g}_{\mathcal{K}_\epsilon} &= \sum_{j=1}^{q}(2k_j+1)j + \frac{1}{2}\sum_{i=1}^{l}m_i(m_i-1)\\
&\geq \sum_{j=1}^{q}(2k_j+1)j\\
&\geq \sum_{j=1}^{q}(2k_j+1)\\
&= \text{The size of the Kronecker part}\\
&= l_2.
\end{aligned}$$

□

Lemma 8.3. $\dim \mathfrak{g}_{C_\epsilon} \geq n$

Proof. Applying Lemma 8.1 and Lemma 8.2, the following inequality can be obtained.

$$\begin{aligned}
\dim \mathfrak{g}_{C_\epsilon} &= n \cdot l_3 + \dim \mathfrak{g}_{\mathcal{J}_\epsilon} + \dim \mathfrak{g}_{\mathcal{K}_\epsilon} + \dim\{P_{ij}\} \\
&\geq n \cdot l_3 + \dim \mathfrak{g}_{\mathcal{J}_\epsilon} + \dim \mathfrak{g}_{\mathcal{K}_\epsilon} \\
&\geq n \cdot l_3 + \frac{3l_1}{2} + l_2 \\
&\geq l_3 + l_1 + l_2 = n.
\end{aligned}$$

□

The following theorem describes generic pencils C_ϵ, i.e. those for which the dimension of $\mathfrak{g}_{C_\epsilon}$ is minimal.

Theorem 8.1. *a) Let n be odd, then the minimal dimension of $\mathfrak{g}_{C_\epsilon}$ is n. Furthermore, $\dim \mathfrak{g}_{C_\epsilon} = n$ iff C_ϵ is one single Kronecker block.*

b) Let n be even, then the minimal dimension of $\mathfrak{g}_{C_\epsilon}$ is $\frac{3n}{2}$. Furthermore, $\dim \mathfrak{g}_{C_\epsilon} = \frac{3n}{2}$ iff C_ϵ is of pure Jordan type and has a single Jordan block for each characteristic number λ.

Proof. a) n is odd.

There are 5 possible choices for C_ϵ when n is odd.

Type 1: C_ϵ is one single Kronecker block of size $n = 2k+1$. Then by using the formula from Dimension Theorem 4.1, we get $\dim \mathfrak{g}_{C_\epsilon} = 2k+1 = n$.

Type 2: C_ϵ is of pure Kronecker type with at least 3 Kronecker blocks ($n = l_2 = \sum_{j=1}^{q}(2k_j+1)$ and $q = 3$, $k_1 \leq k_2 \leq k_3$)

$$\begin{aligned}
\dim \mathfrak{g}_{C_\epsilon} &\geq \sum_{j=1}^{3}(2k_j+1)j + \frac{1}{2}\sum_{i=1}^{l} m_i(m_i-1) \\
&\geq \sum_{j=1}^{3}(2k_j+1)j \\
&= (2k_1+1)\cdot 1 + (2k_2+1)\cdot 2 + (2k_3+1)\cdot 3 \\
&= (2k_1+1) + (2k_2+1) + (2k_3+1) + (2k_2+1) + 2(2k_3+1) \\
&= n + (2k_2+1) + (2k_3+1) + (2k_3+1) \\
&> n + (2k_1+1) + (2k_2+1) + (2k_3+1) \\
&= 2n > n.
\end{aligned}$$

Type 3: C_ϵ contains Jordan blocks and at least one Kronecker block, but no zero blocks.

$(n = l_1 + l_2)$

$$\begin{aligned}
\dim \mathfrak{g}_{C_\epsilon} &= \dim \mathfrak{g}_{\mathcal{J}_\epsilon} + \dim \mathfrak{g}_{\mathcal{K}_\epsilon} + \dim\{P_{ij}\} \\
&> \dim \mathfrak{g}_{\mathcal{J}_\epsilon} + \dim \mathfrak{g}_{\mathcal{K}_\epsilon} \\
&\geq \frac{3l_1}{2} + l_2 \\
&= \frac{l_1}{2} + (l_1 + l_2) \\
&= \frac{l_1}{2} + n \\
&> n.
\end{aligned}$$

Type 4: C_ϵ contains at least one Kronecker block and two zero blocks, but no Jordan blocks. $(n = l_2 + l_3)$

$$\begin{aligned}
\dim \mathfrak{g}_{C_\epsilon} &= n \cdot l_3 + \dim \mathfrak{g}_{\mathcal{K}_\epsilon} \\
&\geq 2n + l_2 \\
&> n.
\end{aligned}$$

Type 5: C_ϵ contains Jordan blocks and at least one zero block, without Kronecker blocks. $(n = l_1 + l_3)$

$$\begin{aligned}
\dim \mathfrak{g}_{C_\epsilon} &= n \cdot l_3 + \dim \mathfrak{g}_{\mathcal{J}_\epsilon} \\
&\geq n + \frac{3l_1}{2} \\
&> n.
\end{aligned}$$

As we can see, the case when C_ϵ contains only one single Kronecker block is the only possibility to obtain $\dim \mathfrak{g}_{C_\epsilon} = n$.

b) n is even.

In this case, there are 5 possible types of C_ϵ.

Type 1: C_ϵ is of pure Jordan type and for each distinct λ_t there is exactly one Jordan block.
$(n = l_1 = \sum_{t=1}^{p} 2k(\lambda_t))$.

$$
\begin{aligned}
\dim \mathfrak{g}_{C_\epsilon} &= \dim \mathfrak{g}_{\mathcal{J}_\epsilon} \\
&= \sum_{t=1}^{p} \left(\sum_{j=1}^{s_t} (4j-1) \cdot k_j(\lambda_t) \right) \\
&(\text{since} \quad s_t = 1) \\
&= \sum_{t=1}^{p} (4-1) k(\lambda_t) \\
&= \frac{3}{2} \sum_{t=1}^{p} 2k(\lambda_t) \\
&= \frac{3n}{2}.
\end{aligned}
$$

Type 2: C_ϵ is of pure Jordan type and contains more than one Jordan block that corresponds to the same λ.

$$
(n = l_1 = \textstyle\sum_{t=1}^{p} \left(\sum_{j=1}^{s_t} 2k_j(\lambda_t) \right), \ s_t \geq 2).
$$

$$
\begin{aligned}
\dim \mathfrak{g}_{C_\epsilon} &= \dim \mathfrak{g}_{\mathcal{J}_\epsilon} \\
&= \sum_{t=1}^{p} \left(\sum_{j=1}^{s_t} (4j-1) \cdot k_j(\lambda_t) \right) \\
&> \sum_{t=1}^{p} \left(\sum_{j=1}^{s_t} 3k_j(\lambda_t) \right) \\
&= \frac{3}{2} \sum_{t=1}^{p} \left(\sum_{j=1}^{s_t} 2k_j(\lambda_t) \right) \\
&= \frac{3n}{2}.
\end{aligned}
$$

Type 3: C_ϵ contains at least 2 zero blocks but no Kronecker blocks.

$$
(n = l_1 + l_3, \ l_3 \geq 2).
$$

$$
\begin{aligned}
\dim \mathfrak{g}_{C_\epsilon} &\geq n \cdot l_3 + \frac{3l_1}{2} \\
&\geq 2n + \frac{3l_1}{2} \\
&\geq 2n \\
&> \frac{3n}{2}.
\end{aligned}
$$

Type 4: C_ϵ contains at least one zero and one Kronecker block.
$(n = l_1 + l_2 + l_3,\ l_3 \geq 1,\ l_2 > 1)$.

$$\begin{aligned}
\dim \mathfrak{g}_{C_\epsilon} &\geq n \cdot l_3 + \frac{3l_1}{2} + l_2 \\
&= n \cdot l_3 + l_1 + l_2 + \frac{l_1}{2} \\
&= n \cdot l_3 + l_1 + l_2 + \frac{1}{2}(n - l_2 - l_3) \\
&= n \cdot l_3 + \frac{n}{2} + l_1 + l_2 - \frac{1}{2}l_2 - \frac{1}{2}l_3 \\
&\geq n \cdot 1 + \frac{n}{2} + l_1 + \frac{l_2}{2} - \frac{1}{2} \\
&= \frac{3n}{2} + l_1 + \frac{l_2}{2} - \frac{1}{2} \\
&> \frac{3n}{2}
\end{aligned}$$

Type 5: C_ϵ contains at least 2 Kronecker blocks but no zero blocks.
$(n = l_1 + l_2,\ l_2 = \sum_{j=1}^{q}(2k_j + 1)$ and $q \geq 2,\ k_2 \geq k_1)$.

$$\begin{aligned}
\dim \mathfrak{g}_{\mathcal{K}_\epsilon} &= \sum_{j=1}^{q}(2k_j + 1)j + \frac{1}{2}\sum_{i=1}^{l} m_i(m_i - 1) \\
&\geq \sum_{j=1}^{q}(2k_j + 1)j \\
&\geq \sum_{j=1}^{2}(2k_j + 1)j \\
&= (2k_1 + 1) \cdot 1 + (2k_2 + 1) \cdot 2 \\
&= (2k_1 + 1) + \frac{1}{2}(2k_2 + 1) + \frac{3}{2}(2k_2 + 1) \\
&> (2k_1 + 1) + \frac{1}{2}(2k_1 + 1) + \frac{3}{2}(2k_2 + 1) \\
&= \frac{3}{2}(2k_1 + 1) + \frac{3}{2}(2k_2 + 1) \\
&= \frac{3l_2}{2}
\end{aligned}$$

$$\begin{aligned}
\dim \mathfrak{g}_{C_\epsilon} &\geq \dim \mathfrak{g}_{\mathcal{J}_\epsilon} + \dim \mathfrak{g}_{\mathcal{K}_\epsilon} \\
&> \frac{3l_1}{2} + \frac{3l_2}{2} \\
&= \frac{3}{2}(l_1 + l_2) \\
&= \frac{3n}{2}
\end{aligned}$$

As we have shown above the minimum dimension of $\mathfrak{g}_{C_\epsilon}$ is $\frac{3n}{2}$ for even n. This minimum dimension may only occur in the Type 1, where C_ϵ contains exactly one Jordan block for each distinct characteristic number λ. $\square$

9 Algebraic properties of $\mathfrak{g}_{C_\epsilon}$

Some other algebraic properties can be further analyzed from the structure of $\mathfrak{g}_{C_\epsilon}$ with respect to the Jordan-Kronecker decomposition.

Property 1. *If a pencil C_ϵ contains Jordan blocks, then $\mathfrak{g}_{C_\epsilon}$ is not solvable.*

Proof. Let C_ϵ be a single Jordan block i.e.

$$C_\epsilon = \left(\begin{array}{ccc|ccc}
 & & & \lambda+\epsilon & 1 & \\
 & & & & \ddots & \ddots 1 \\
 & & & & & \lambda+\epsilon \\
\hline
-\lambda-\epsilon & & & & & \\
-1 & \ddots & & & & \\
 & \ddots -1 & -\lambda-\epsilon & & &
\end{array}\right), \text{ so we have } \mathfrak{g}_{C_\epsilon} = \left(\begin{array}{c}
\begin{array}{cccccc} e_1 & \dots & e_k & f_1 & \dots & f_k \end{array} \\
\boxed{\begin{array}{ccc|ccc}
a_1 & \dots & a_k & b_k & \dots & b_1 \\
 & \ddots & \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \\
 & & a_1 & b_1 & & \\
\hline
 & & c_1 & -a_1 & & \\
 & \cdot^{\cdot^{\cdot}} & \vdots & \vdots & \ddots & \\
c_1 & \dots & c_k & -a_k & \dots & -a_1
\end{array}}
\end{array}\right).$$

After the following change of basis:

$$e_1 e_2 \dots e_k f_1 f_2 \dots f_k \to e_1 f_k e_2 f_{k-1} \dots e_k f_1$$

we have

$$\mathfrak{g}_{C_\epsilon} \simeq \begin{pmatrix} \boxed{\begin{matrix} a_1 & b_1 \\ c_1 & -a_1 \end{matrix}} & \boxed{\begin{matrix} a_2 & b_2 \\ c_2 & -a_2 \end{matrix}} & \cdots & \boxed{\begin{matrix} a_k & b_k \\ c_k & -a_k \end{matrix}} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \boxed{\begin{matrix} a_2 & b_2 \\ c_2 & -a_2 \end{matrix}} \\ & & & \boxed{\begin{matrix} a_1 & b_1 \\ c_1 & -a_1 \end{matrix}} \end{pmatrix} = sl(2,\mathbb{R}) \otimes \mathbb{P}_{k-1},$$

where $\mathbb{P}_{k-1}$ be an algebra of truncated polynomials of degree $\leq k-1$:

$$\mathbb{P}_{k-1} = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_{k-1}x^{k-1}\} \text{ with } x^i x^j = \begin{cases} x^{i+j}, & i+j \leq k-1 \\ 0, & i+j > k-1 \end{cases}$$

So we can see $\mathfrak{g}_{C_\epsilon}$ contains a semisimple Lie subalgebra:

$$\begin{pmatrix} \boxed{\begin{matrix} a_1 & b_1 \\ c_1 & -a_1 \end{matrix}} & & \\ & \ddots & \\ & & \boxed{\begin{matrix} a_1 & b_1 \\ c_1 & -a_1 \end{matrix}} \end{pmatrix} = sl(2,\mathbb{R}),$$

which means that $\mathfrak{g}_{C_\epsilon}$ is not solvable. More generally, for any pencil (C_ϵ) that contains Jordan blocks, the corresponding Lie algebra $(\mathfrak{g}_{C_\epsilon})$ contains a semisimple Lie subalgebra (as above), which means $\mathfrak{g}_{C_\epsilon}$ is not solvable.

$\square$

Property 2. *Consider C_ϵ be a single Kronecker block, then $\mathfrak{g}_{C_\epsilon}$ is solvable but not nilpotent.*

Proof. Consider a single Kronecker block C_ϵ and the corresponding Lie algebra $\mathfrak{g}_{C_\epsilon}$:

$$C_\epsilon = \left(\begin{array}{ccc|ccc} & & & \epsilon & 1 & \\ & & & & \ddots & \ddots \\ & & & & & \epsilon \quad 1 \\ \hline -\epsilon & & & & & \\ -1 & \ddots & & & & \\ & \ddots & -\epsilon & & & \\ & & -1 & & & \end{array}\right), \text{ and } \mathfrak{g}_{C_\epsilon} = \left(\begin{array}{ccc|ccccc} a & & & b_1 & b_2 & \cdots & b_k & b_{k+1} \\ & \ddots & & b_2 & & \iddots & \iddots & \vdots \\ & & \ddots & \vdots & \iddots & \iddots & & \vdots \\ & & a & b_k & b_{k+1} & \cdots & \cdots & b_{k+k} \\ \hline & & & -a & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & \ddots & \\ & & & & & & & -a \end{array}\right)$$

We can see $\mathfrak{g}_{C_\epsilon}$ is a Lie subalgebra of the Lie algebra of upper triangular matrices, and therefore is solvable. Moreover, we have

$$
\begin{aligned}
\mathfrak{g}_{C_\epsilon}^{(1)} &= [\mathfrak{g}_{C_\epsilon}, \mathfrak{g}_{C_\epsilon}] = \left(\begin{array}{c|c} & * \\ \hline & \end{array}\right) \\
\mathfrak{g}_{C_\epsilon}^{(2)} &= [\mathfrak{g}_{C_\epsilon}^{(1)}, \mathfrak{g}_{C_\epsilon}^{(1)}] = \{0\},
\end{aligned}
$$

where $*$ denotes the entries in the block. Thus, $\mathfrak{g}_{C_\epsilon}$ is two-step solvable.

However, $\mathfrak{g}_{C_\epsilon}$ is not nilpotent, this can be checked by the direct computation. Indeed, we have

$$
\begin{aligned}
\mathfrak{g}'_{C_\epsilon} &= [\mathfrak{g}_{C_\epsilon}, \mathfrak{g}_{C_\epsilon}] = \left(\begin{array}{c|c} & * \\ \hline & \end{array}\right) \\
\mathfrak{g}''_{C_\epsilon} &= [\mathfrak{g}'_{C_\epsilon}, \mathfrak{g}_{C_\epsilon}] = \left(\begin{array}{c|c} & * \\ \hline & \end{array}\right) = \mathfrak{g}'_{C_\epsilon}.
\end{aligned}
$$

Thus, $\mathfrak{g}_{C_\epsilon}$ is not nilpotent. $\square$

Property 3. *If C_ϵ contains at least 2 equal sized Kronecker blocks, then $\mathfrak{g}_{C_\epsilon}$ is not solvable.*

Proof. We first consider a pencil C_ϵ consisting of 2 Kronecker blocks of the same size and the corresponding Lie algebra $\mathfrak{g}_{C_\epsilon}$:

$$
C_\epsilon = \left(\begin{array}{c|c}
\boxed{\begin{array}{ccc|ccc}
 & & & \epsilon & 1 & \\
 & & & & \ddots & \ddots \\
 & & & & \epsilon & 1 \\ \hline
-\epsilon & & & & & \\
-1 & \ddots & & & & \\
 & \ddots & -\epsilon & & & \\
 & & -1 & & &
\end{array}} & \\
 & \boxed{\begin{array}{ccc|ccc}
 & & & \epsilon & 1 & \\
 & & & & \ddots & \ddots \\
 & & & & \epsilon & 1 \\ \hline
-\epsilon & & & & & \\
-1 & \ddots & & & & \\
 & \ddots & -\epsilon & & & \\
 & & -1 & & &
\end{array}}
\end{array}\right),
$$

and

$$
\mathfrak{g}_{C_\epsilon} = \begin{array}{c}
\begin{array}{cccc} \overbrace{\qquad\qquad}^{e_1} & \overbrace{\qquad\qquad\qquad}^{e_2} & \overbrace{\qquad\qquad}^{e_3} & \overbrace{\qquad\qquad\qquad}^{e_4} \end{array} \\
\left(\begin{array}{c|c}
\begin{array}{ccc|ccc}
a_1 & & & b_1 & \cdots & b_{k+1} \\
 & \ddots & & \vdots & \iddots\iddots & \vdots \\
 & & a_1 & b_k & \cdots & b_{2k} \\ \hline
 & & & -a_1 & & \\
 & & & & \ddots & \\
 & & & & & -a_1
\end{array} &
\begin{array}{ccc|ccc}
a_3 & & & d_1 & \cdots & d_{k+1} \\
 & \ddots & & \vdots & \iddots\iddots & \vdots \\
 & & a_3 & d_k & \cdots & d_{2k} \\ \hline
 & & & -a_4 & & \\
 & & & & \ddots & \\
 & & & & & -a_4
\end{array} \\ \hline
\begin{array}{ccc|ccc}
a_4 & & & e_1 & \cdots & e_{k+1} \\
 & \ddots & & \vdots & \iddots\iddots & \vdots \\
 & & a_4 & e_k & \cdots & e_{2k} \\ \hline
 & & & -a_3 & & \\
 & & & & \ddots & \\
 & & & & & -a_3
\end{array} &
\begin{array}{ccc|ccc}
a_2 & & & c_1 & \cdots & c_{k+1} \\
 & \ddots & & \vdots & \iddots\iddots & \vdots \\
 & & a_2 & c_k & \cdots & c_{2k} \\ \hline
 & & & -a_2 & & \\
 & & & & \ddots & \\
 & & & & & -a_2
\end{array}
\end{array}\right)
\end{array}.
$$

Again, the change of basis: $e_1e_2e_3e_4 \to e_1e_3e_2e_4$, give us

$$\mathfrak{g}_{C_\epsilon} \simeq \left(\begin{array}{cc} \boxed{\begin{array}{ccc|ccc} a_1 & & & a_3 & & \\ & \ddots & & & \ddots & \\ & & a_1 & & & a_3 \\ \hline a_4 & & & a_2 & & \\ & \ddots & & & \ddots & \\ & & a_4 & & & a_2 \end{array}} & *** \\ & \boxed{\begin{array}{ccc|ccc} -a_1 & & & -a_4 & & \\ & \ddots & & & \ddots & \\ & & -a_1 & & & -a_4 \\ \hline -a_3 & & & -a_2 & & \\ & \ddots & & & \ddots & \\ & & -a_3 & & & -a_2 \end{array}} \end{array}\right) \supset gl(2,\mathbb{R}) \simeq sl(2,\mathbb{R}) \oplus \mathbb{R}$$

which means that $\mathfrak{g}_{C_\epsilon}$ contains a semisimple Lie subalgebra isomorphic to $sl(2,\mathbb{R})$. It is easy to see that for any pencil C_ϵ containing at least 2 equal sized Kronecker blocks, its Lie algebra $\mathfrak{g}_{C_\epsilon}$ always contains the above semisimple Lie subalgebra. Therefore $\mathfrak{g}_{C_\epsilon}$ is not solvable.

□

The above three properties lead us to the following conclusion:

Theorem 9.1. *$\mathfrak{g}_{C_\epsilon}$ is solvable $\Leftrightarrow$ C_ϵ consists of Kronecker blocks with distinct sizes.*

Proof. Properties 1 and 3 tell us that for $\mathfrak{g}_{C_\epsilon}$ to be solvable, the pencil C_ϵ should contain neither Jordan blocks nor equal sized Kronecker blocks. Thus the only possible pencil C_ϵ is that with distinct sized Kronecker blocks. Therefore we have:

$\mathfrak{g}_{C_\epsilon}$ is solvable $\Rightarrow$ C_ϵ consists of Kronecker blocks with distinct sizes.

On the other hand, we need to check the inverse statement. To see this, we shall first consider a pencil C_ϵ that consists of 2 Kronecker blocks of different sizes.

$$C_\epsilon = \left(\begin{array}{cc} \boxed{\begin{array}{ccc|ccc} & & & \epsilon & 1 & \\ & & & & \ddots & \ddots \\ & & & & \epsilon & 1 \\ \hline -\epsilon & & & & & \\ -1 & \ddots & & & & \\ & \ddots & -\epsilon & & & \\ & & -1 & & & \end{array}} & \\ & \boxed{\begin{array}{ccc|ccc} & & & \epsilon & 1 & \\ & & & & \ddots \epsilon & \ddots 1 \\ \hline -\epsilon & & & & & \\ -1 & \ddots -\epsilon & & & & \\ & \ddots -1 & & & & \end{array}} \end{array}\right)$$

$$
\text{and } \mathfrak{g}_{C_\epsilon} = \begin{array}{c}
\begin{array}{cccc} \overbrace{\qquad\qquad\qquad}^{e_1} & \overbrace{\qquad\qquad\qquad}^{e_2} & \overbrace{\qquad\qquad}^{e_3} & \overbrace{\qquad\qquad\qquad}^{e_4}\end{array}\\
\left(\begin{array}{ccc|ccc|ccc|ccc}
a_1 & & & b_1 & \cdots & b_{k_1+1} & & & & d_1 & \cdots & d_{k_2+1}\\
& \ddots & & \vdots & \ddots\ddots & \vdots & & & & \vdots & \ddots\ddots & \vdots\\
& & a_1 & b_{k_1} & \cdots & b_{2k_1} & & & & d_{k_1} & \cdots & d_{k_1+k_2}\\
\hline
& & & -a_1 & & & & & & -e_1 & & \\
& & & & \ddots & & & & & -e_{k_1-k_2+1} & \ddots & -e_1\\
& & & & & -a_1 & & & & & \ddots & -e_{k_1-k_2+1}\\
\hline
e_1 \ldots e_{k_1-k_2+1} & & & d_1 & \cdots & d_{k_1+1} & a_2 & & & c_1 & \cdots & c_{k_2+1}\\
\ddots & \ddots & & \vdots & \iddots\iddots & \vdots & & \ddots & & \vdots & \ddots\ddots & \vdots\\
& & e_1 \ldots e_{k_1-k_2+1} & d_{k_2} & \cdots & d_{k_1+k_2} & & & a_2 & c_{k_2} & \cdots & c_{2k_2}\\
\hline
& & & & & & & & & -a_2 & & \\
& & & & & & & & & & \ddots & \\
& & & & & & & & & & & -a_2
\end{array}\right)
\end{array}.
$$

After change of basis: $e_1e_2e_3e_4 \to e_3e_1e_2e_4$, the algebra $\mathfrak{g}_{C_\epsilon}$ can be reduced to a subalgebra of the Lie algebra of upper triangular matrices:

$$
\mathfrak{g}_{C_\epsilon} \simeq \left(\begin{array}{ccc|ccc|ccc|ccc}
a_2 & & & e_1 \ldots e_{k_1-k_2+1} & & & d_1 & \cdots & d_{k_1+1} & c_1 & \cdots & c_{k_2+1}\\
& \ddots & & \ddots & \ddots & & \vdots & \iddots\iddots & \vdots & \vdots & \iddots\iddots & \vdots\\
& & a_2 & & & e_1 \ldots e_{k_1-k_2+1} & d_{k_2} & \cdots & d_{k_1+k_2} & c_{k_2} & \cdots & c_{2k_2}\\
\hline
& & & a_1 & & & b_1 & \cdots & b_{k_1+1} & d_1 & \cdots & d_{k_2+1}\\
& & & & \ddots & & \vdots & \iddots\iddots & \vdots & \vdots & \iddots\iddots & \vdots\\
& & & & & a_1 & b_{k_1} & \cdots & b_{2k_1} & d_{k_1} & \cdots & d_{k_1+k_2}\\
\hline
& & & & & & -a_1 & & & -e_1 & & \\
& & & & & & & \ddots & & -e_{k_1-k_2+1} & \ddots & -e_1\\
& & & & & & & & -a_1 & & \ddots & -e_{k_1-k_2+1}\\
\hline
& & & & & & & & & -a_2 & & \\
& & & & & & & & & & \ddots & \\
& & & & & & & & & & & -a_2
\end{array}\right)
$$

Similarly, for a pencil C_ϵ that contains n distinct sized Kronecker blocks, we change the basis in the following way: (for $i \in 1, \ldots, n$)

$$
e_1e_2\ldots e_{2n} \to \underbrace{e_{2n-1}e_{2n-3}\ldots e_1}_{e_{2n-(2i-1)}}\underbrace{e_2e_4\ldots e_{2n}}_{e_{2i}}
$$

to reduce $\mathfrak{g}_{C_\epsilon}$ to an upper triangular form which means that $\mathfrak{g}_{C_\epsilon}$ is solvable. $\square$

Recall Property 1, we know $\mathfrak{g}_{\mathcal{J}_\epsilon}$ is not solvable when we have the pencil $\mathcal{J}_\epsilon$ is in the Jordan case. However, we can still generate the following propositions on $\mathfrak{g}_{\mathcal{J}_\epsilon}$ when $\mathcal{J}_\epsilon$ consist of pure Jordan blocks.

Proposition 1. *Let $\mathcal{J}_\epsilon$ be a single Jordan block, then the semisimple Levi subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\mathcal{J}_\epsilon}$ is isomorphic to $sl(2, \mathbb{R})$.*

The sketch of proof: (see the proof of Property 1)

Proposition 2. *Let $\mathcal{J}_\epsilon$ consist of n Jordan blocks, and all blocks have distinct characteristic values ($\lambda_i \neq \lambda_j$, $i, j \in 1, \ldots, n$). Then the semisimple Levi subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\mathcal{J}_\epsilon}$ is isomorphic to $\underbrace{sl(2,\mathbb{R}) \oplus \cdots \oplus sl(2,\mathbb{R})}_{n}$.*

Proof. Let $\mathcal{J}_\epsilon$ consist of 2 Jordan blocks of size $2k_1$ and $2k_2$ with distinct characteristic values λ and μ respectively.

$$\mathcal{J}_\epsilon = \begin{pmatrix} \boxed{\begin{array}{c|c} & \begin{matrix} \lambda+\epsilon & 1 & \\ & \ddots & \ddots 1 \\ & & \lambda+\epsilon \end{matrix} \\ \hline \begin{matrix} -\lambda-\epsilon & & \\ -1 & \ddots & \\ & \ddots -1 & -\lambda-\epsilon \end{matrix} & \end{array}} & \\ & \boxed{\begin{array}{c|c} & \begin{matrix} \mu+\epsilon & 1 & \\ & \ddots & \ddots 1 \\ & & \mu+\epsilon \end{matrix} \\ \hline \begin{matrix} -\mu-\epsilon & & \\ -1 & \ddots & \\ & \ddots -1 & -\mu-\epsilon \end{matrix} & \end{array}} \end{pmatrix}$$

According to Property 1, we know the algebraic structure of $\mathfrak{g}_{\mathcal{J}_\epsilon}$ w.r.t each characteristic value. Namely, we have

$$\mathfrak{g}_{\mathcal{J}_\epsilon} \simeq \begin{pmatrix} \boxed{\begin{matrix} \boxed{\begin{matrix} a_1 & b_1 \\ c_1 & -a_1 \end{matrix}} & \cdots & \boxed{\begin{matrix} a_{k_1} & b_{k_1} \\ c_{k_1} & -a_{k_1} \end{matrix}} \\ & \ddots & \vdots \\ & & \boxed{\begin{matrix} a_1 & b_1 \\ c_1 & -a_1 \end{matrix}} \end{matrix}} & \\ & \boxed{\begin{matrix} \boxed{\begin{matrix} d_1 & e_1 \\ f_1 & -d_1 \end{matrix}} & \cdots & \boxed{\begin{matrix} d_{k_2} & e_{k_2} \\ f_{k_2} & -d_{k_2} \end{matrix}} \\ & \ddots & \vdots \\ & & \boxed{\begin{matrix} d_1 & e_1 \\ f_1 & -d_1 \end{matrix}} \end{matrix}} \end{pmatrix} = sl(2,\mathbb{R}) \otimes \mathbb{P}_{k_1-1} \bigoplus sl(2,\mathbb{R}) \otimes \mathbb{P}_{k_2-1}$$

The semisimple Levi subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\mathcal{J}_\epsilon}$ is isomorphic to $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$.

In general, if $\mathcal{J}_\epsilon$ consists of n Jordan blocks of size $2k_1, \ldots, 2k_n$ with distinct characteristic value $\lambda_1, \ldots, \lambda_n$ ($\lambda_i \neq \lambda_j$, $i, j \in 1, \ldots, n$) respectively, and $k_1 \geq \cdots \geq k_n$, we have $\mathfrak{g}_{\mathcal{J}_\epsilon} \simeq \bigoplus_i^n sl(2,\mathbb{R}) \otimes \mathbb{P}_{k_i-1}$ and $\mathfrak{h} \simeq \underbrace{sl(2,\mathbb{R}) \oplus \cdots \oplus sl(2,\mathbb{R})}_{n}$. □

Proposition 3. *$\mathcal{J}_\epsilon$ consist of n distinct sized Jordan blocks, which all have the same characteristic value (say λ), then the semisimple Lie subalgebra ($\mathfrak{h} \subset \mathfrak{g}_{\mathcal{J}_\epsilon}$) is isomorphic to $\underbrace{sl(2,\mathbb{R}) \oplus \cdots \oplus sl(2,\mathbb{R})}_{n}$.*

Proof. First, we consider an example:

$$
\mathcal{J}_\epsilon = \left(\begin{array}{cc}
\boxed{\begin{array}{ccc|ccc}
 & & & \lambda+\epsilon & 1 & \\
 & & & & \lambda+\epsilon & 1 \\
 & & & & & \lambda+\epsilon \\
\hline
-\lambda-\epsilon & & & & & \\
-1 & -\lambda-\epsilon & & & & \\
 & -1 & -\lambda-\epsilon & & &
\end{array}} & \\
 & \boxed{\begin{array}{cc|cc}
 & & \lambda+\epsilon & 1 \\
 & & & \lambda+\epsilon \\
\hline
-\lambda-\epsilon & & & \\
-1 & -\lambda-\epsilon & &
\end{array}}
\end{array}\right)
$$

$$
\text{and } \mathfrak{g}_{\mathcal{J}_\epsilon} \simeq
\begin{array}{c}
\begin{array}{ccccc} \overbrace{\qquad\qquad}^{e_1} & \overbrace{\qquad\qquad}^{e_2} & \overbrace{\qquad\qquad}^{e_3} & \overbrace{\qquad\qquad}^{f_1} & \overbrace{\qquad\qquad}^{f_2} \end{array} \\
\left(\begin{array}{ccc|cc}
\boxed{\begin{matrix} a_1 & b_1 \\ c_1 & -a_1 \end{matrix}} & \boxed{\begin{matrix} a_2 & b_2 \\ c_2 & -a_2 \end{matrix}} & \boxed{\begin{matrix} a_3 & b_3 \\ c_3 & -a_3 \end{matrix}} & \boxed{\begin{matrix} m_1 & h_1 \\ p_1 & -g_1 \end{matrix}} & \boxed{\begin{matrix} m_2 & h_2 \\ p_2 & -g_2 \end{matrix}} \\
 & \boxed{\begin{matrix} a_1 & b_1 \\ c_1 & -a_1 \end{matrix}} & \boxed{\begin{matrix} a_2 & b_2 \\ c_2 & -a_2 \end{matrix}} & & \boxed{\begin{matrix} m_1 & h_1 \\ p_1 & -g_1 \end{matrix}} \\
 & & \boxed{\begin{matrix} a_1 & b_1 \\ c_1 & -a_1 \end{matrix}} & & \\
\hline
 & \boxed{\begin{matrix} g_1 & h_1 \\ p_1 & -m_1 \end{matrix}} & \boxed{\begin{matrix} g_2 & h_2 \\ p_2 & -m_2 \end{matrix}} & \boxed{\begin{matrix} d_1 & e_1 \\ f_1 & -d_1 \end{matrix}} & \boxed{\begin{matrix} d_2 & e_2 \\ f_2 & -d_2 \end{matrix}} \\
 & & \boxed{\begin{matrix} g_1 & h_1 \\ p_1 & -m_1 \end{matrix}} & & \boxed{\begin{matrix} d_1 & e_1 \\ f_1 & -d_1 \end{matrix}}
\end{array}\right)
\end{array}.
$$

Again, after the change of basis: $e_1e_2e_3e_4f_1f_2 \to e_1f_1e_2f_2e_3$, we get

$$
\mathfrak{g}_{\mathcal{J}_\epsilon} \simeq
\left(\begin{array}{ccccc}
\boxed{\begin{matrix} a_1 & b_1 \\ c_1 & -a_1 \end{matrix}} & \boxed{\begin{matrix} m_1 & h_1 \\ p_1 & -g_1 \end{matrix}} & \boxed{\begin{matrix} a_2 & b_2 \\ c_2 & -a_2 \end{matrix}} & \boxed{\begin{matrix} m_2 & h_2 \\ p_2 & -g_2 \end{matrix}} & \boxed{\begin{matrix} a_3 & b_3 \\ c_3 & -a_3 \end{matrix}} \\
 & \boxed{\begin{matrix} d_1 & e_1 \\ f_1 & -d_1 \end{matrix}} & \boxed{\begin{matrix} g_1 & h_1 \\ p_1 & -m_1 \end{matrix}} & \boxed{\begin{matrix} d_2 & e_2 \\ f_2 & -d_1 \end{matrix}} & \boxed{\begin{matrix} g_2 & h_2 \\ p_2 & -m_2 \end{matrix}} \\
 & & \boxed{\begin{matrix} a_1 & b_1 \\ c_1 & -a_1 \end{matrix}} & \boxed{\begin{matrix} m_1 & h_1 \\ p_1 & -g_1 \end{matrix}} & \boxed{\begin{matrix} a_2 & b_2 \\ c_2 & -a_2 \end{matrix}} \\
 & & & \boxed{\begin{matrix} d_1 & e_1 \\ f_1 & -d_1 \end{matrix}} & \boxed{\begin{matrix} g_1 & h_1 \\ p_1 & -m_1 \end{matrix}} \\
 & & & & \boxed{\begin{matrix} a_1 & b_1 \\ c_1 & -a_1 \end{matrix}}
\end{array}\right)
$$

It shows

$$\mathfrak{h} \simeq \begin{pmatrix} \boxed{\begin{matrix} a_1 & b_1 \\ c_1 & -a_1 \end{matrix}} & & & & \\ & \boxed{\begin{matrix} d_1 & e_1 \\ f_1 & -d_1 \end{matrix}} & & & \\ & & \boxed{\begin{matrix} a_1 & b_1 \\ c_1 & -a_1 \end{matrix}} & & \\ & & & \boxed{\begin{matrix} d_1 & e_1 \\ f_1 & -d_1 \end{matrix}} & \\ & & & & \boxed{\begin{matrix} a_1 & b_1 \\ c_1 & -a_1 \end{matrix}} \end{pmatrix} = sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$$

More generally, for $\mathcal{J}_\epsilon$ consisting of n distinct sized Jordan blocks with the same characteristic value, after changing basis in a similar way, its Lie algebra $\mathfrak{g}_{\mathcal{J}_\epsilon}$ can always be arranged into a Lie subalgebra of blocked upper triangular matrices, which include a semisimple Lie subalgebra $\mathfrak{h} \simeq \underbrace{sl(2, \mathbb{R}) \oplus \cdots \oplus sl(2, \mathbb{R})}_{n}$ as a Levi subalgebra. □

Proposition 4. *Let $\mathcal{J}_\epsilon$ consist of n Jordan blocks of the same size, which all have the same characteristic value (say λ), then the semisimple Levi subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\mathcal{J}_\epsilon}$ is isomorphic to $sp(2 \times n, \mathbb{R})$.*

Proof. Take $\mathcal{J}_\epsilon$ consisting of n equal sized (size $2k$) Jordan blocks, which have the same characteristic value (λ) i.e.

$$\mathcal{J}_\epsilon = \begin{pmatrix} \boxed{\begin{array}{c|c} & \begin{matrix} \lambda+\epsilon & 1 & \\ & \ddots & \ddots\ 1 \\ & & \lambda+\epsilon \end{matrix} \\ \hline \begin{matrix} -\lambda-\epsilon & & \\ -1 & \ddots & \\ & \ddots & -1\ -\lambda-\epsilon \end{matrix} & \end{array}} & \\ & \ddots & \\ & & \boxed{\begin{array}{c|c} & \begin{matrix} \lambda+\epsilon & 1 & \\ & \ddots & \ddots\ 1 \\ & & \lambda+\epsilon \end{matrix} \\ \hline \begin{matrix} -\lambda-\epsilon & & \\ -1 & \ddots & \\ & \ddots & -1\ -\lambda-\epsilon \end{matrix} & \end{array}} \end{pmatrix}$$

then, after an appropriate change of basis, we have

$$\mathfrak{g}_{\mathcal{J}_\epsilon} \simeq \begin{pmatrix} \boxed{\begin{array}{ccc|ccc} a_1 & \cdots & a_{1n} & b_1 & \cdots & b_n \\ \vdots & \ddots & \vdots & \vdots & \iddots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_n & \cdots & b_{2n-1} \\ \hline c_1 & \cdots & c_n & -a_1 & \cdots & -a_{n1} \\ \vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\ c_n & \cdots & c_{2n-1} & -a_{1n} & \cdots & -a_{nn} \end{array}} & \cdots & \boxed{\begin{array}{ccc|ccc} m_1 & \cdots & m_{1n} & p_1 & \cdots & p_n \\ \vdots & \ddots & \vdots & \vdots & \iddots & \vdots \\ m_{n1} & \cdots & m_{nn} & p_n & \cdots & p_{2n-1} \\ \hline q_1 & \cdots & q_n & -m_1 & \cdots & -m_{n1} \\ \vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\ q_n & \cdots & q_{2n-1} & -m_{1n} & \cdots & -m_{nn} \end{array}} \\ & \ddots & \vdots \\ & & \boxed{\begin{array}{ccc|ccc} a_1 & \cdots & a_{1n} & b_1 & \cdots & b_n \\ \vdots & \ddots & \vdots & \vdots & \iddots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_n & \cdots & b_{2n-1} \\ \hline c_1 & \cdots & c_n & -a_1 & \cdots & -a_{n1} \\ \vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\ c_n & \cdots & c_{2n-1} & -a_{1n} & \cdots & -a_{nn} \end{array}} \end{pmatrix} = sp(2\times n, \mathbb{R}) \otimes \mathbb{P}_{k-1}$$

It is clear that the Levi subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\mathcal{J}_\epsilon}$ is isomorphic to $sp(2 \times n, \mathbb{R})$. $\square$

According to the above propositions we come to the following conclusion about the Levi subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\mathcal{J}_\epsilon}$ in the case of a general pure Jordan type pencil $(\mathcal{J}_\epsilon)$.

Theorem 9.2. *Suppose $\mathcal{J}_\epsilon$ is a pencil consisting of Jordan blocks only. Let n be the number of distinct characteristic numbers $\lambda_1, \dots, \lambda_n$, and for each characteristic number λ_t we have $s_i = s_i(\lambda_t)$ Jordan blocks of size $k_i(\lambda_t)$, $i = 1, \dots, m_t$:*

$$\underbrace{k_1(\lambda_t)\dots k_1(\lambda_t)}_{s_1}\underbrace{k_2(\lambda_t)\dots k_2(\lambda_t)}_{s_2}\dots\underbrace{k_i(\lambda_t)\dots k_i(\lambda_t)}_{s_i}\dots\underbrace{k_{m_t}(\lambda_t)\dots k_{m_t}(\lambda_t)}_{s_{m_t}}.$$

Then the semisimple Levi subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\mathcal{J}_\epsilon}$ is isomorphic to

$$\bigoplus_{t=1}^{n}\bigoplus_{i=1}^{m_t} sp(2 \times s_i(\lambda_t), \mathbb{R}).$$

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